DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCES- MATHEMATICS

SEMESTER -IV

GENERAL THEORY OF INTEGRATION

DEMATH4ELEC7

BLOCK-2

UNIVERSITY OF NORTH BENGAL

Postal Address:

The Registrar,

University of North Bengal,

Raja Rammohunpur,

P.O.-N.B.U.,Dist-Darjeeling,

West Bengal, Pin-734013,

India.

Phone: (O) +91 0353-2776331/2699008

Fax:(0353) 2776313, 2699001

Email: regnbu@sancharnet.in ; regnbu@nbu.ac.in

Wesbsite: www.nbu.ac.in

First Published in 2019

FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

GENERAL THEORY OF INTEGRATION

BLOCK-1

BLOCK-2

BLOCK-2 GENERAL THEORY OF INTEGRATION

Introduction to Block-2

The Elements of Lebesgue Measure is descended from class notes written to acquaint the reader with the theory of Lebesgue measure in the space Rp. While it is easy to find good treatments of the case P = 1, the case p > 1 is not quite as simple and is much less frequently discussed. The main ideas of Lebesgue measure, absolute are presented in detail in Units 8,9 although some relatively easy remarks are left to the reader as exercises. And these units venture into the topic of non measurable sets and round out the subject.

There are many expositions of the Lebesgue integral from various points of view, but I believe that the abstract measure space approach used here strikes directly towards the most important results: the convergence theorems. Further, this approach is particularly well suited for students of probability and statistics, as well as students of analysis. Since the book is intended as an introduction, I do not follow all of the avenues that are encountered. Even in calculus courses, one needs to extend the integral by defining "improper integrals", either because the integrand has a singularity, or because the interval of integration is infinite. In addition, by taking pointwise limits of Riemann integrable functions, one quickly encounters functions that are no longer Riemann integrable. Even when one requires uniform convergence, there are problems on infinite intervals. In block-2 we will learn and understand about Measures, Modes of converges, Application to calculus, Improper Integrals, Substitution theorems, Absolute Continuity, Mapping properties of AC functions.

UNIT-8 MEASURES

STRUCTURE

8.0 OBJECTIVE

In this unit we will learn and understand about Lebesgue integration, Definitions related to measures, Theorems, Examples and The Riemann Darboux approach.

8.1 INTRODUCTION

In mathematics, the integral of a non-negative function of a single variable can be regarded, in the simplest case, as the area between the graph of that function and the *x*-axis. The Lebesgue integral extends the integral to a larger class of functions. It also extends the domains on which these functions can be defined.

Long before the 20th century, mathematicians already understood that for non-negative functions with a smooth enough graph—such as continuous functions on closed bounded intervals—the *area under the curve* could be defined as the integral, and computed using approximation techniques on the region by polygons. However, as the need to consider more irregular functions arose—e.g., as a result of the limiting processes of mathematical analysis and the mathematical theory of probability—it became clear that more careful approximation techniques were needed to define a suitable integral. Also, one might wish to integrate on spaces more general than the real line. The Lebesgue integral provides abstractions needed to do this important job.

The Lebesgue integral plays an important role in probability theory, real analysis, and many other fields in the mathematical sciences. It is named after Henri Lebesgue (1875–1941), who introduced the integral (Lebesgue 1904). It is also a pivotal part of the axiomatic theory of probability.

The term *Lebesgue integration* can mean either the general theory of integration of a function with respect to a general measure, as introduced by Lebesgue, or the specific case of integration of a function defined on a sub-domain of the real line with respect to the Lebesgue measure.

8.2 LEBESGUE INTEGRATION

The integral of a positive function *f* between limits *a* and *b* can be interpreted as the area under the graph of *f*. This is easy to understand for familiar functions such as polynomials, but what does it mean for more exotic functions? In general, for which class of functions does "area under the curve" make sense? The answer to this question has great theoretical and practical importance.

As part of a general movement toward rigor in mathematics in the nineteenth century, mathematicians attempted to put integral calculus on a firm foundation. The Riemann integral—proposed by Bernhard Riemann (1826–1866)—is a broadly successful attempt to provide such a foundation. Riemann's definition starts with the construction of a sequence of easily calculated areas that converge to the integral of a given function. This definition is successful in the sense that it gives the expected answer for many already-solved problems, and gives useful results for many other problems.

However, Riemann integration does not interact well with taking limits of sequences of functions, making such limiting processes difficult to analyze. This is important, for instance, in the study of Fourier series, Fourier transforms, and other topics. The Lebesgue integral is better able to describe how and when it is possible to take limits under the integral sign (via the powerful monotone convergence theorem and dominated convergence theorem).

While the Riemann integral considers the area under a curve as made out of vertical rectangles, the Lebesgue definition considers horizontal slabs that are not necessarily just rectangles, and so it is more flexible. For this reason, the Lebesgue definition makes it possible to calculate integrals for a broader class of functions. For example, the Dirichlet function, which is 0 where its argument is irrational and 1 otherwise, has a Lebesgue integral, but does not have a Riemann integral. Furthermore, the Lebesgue integral of this function is zero, which agrees with the intuition that when picking a real number uniformly at random from the unit interval, the probability of picking a rational number should be zero.

Lebesgue summarized his approach to integration in a letter to Paul Montel:

I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.

The insight is that one should be able to rearrange the values of a function freely, while preserving the value of the integral. This process of rearrangement can convert a very pathological function into one that is "nice" from the point of view of integration, and thus let such pathological functions be integrated.

Intuitive interpretation.

To get some intuition about the different approaches to integration, let us imagine that we want to find a mountain's volume (above sea level).

8.3 THE RIEMANN–DARBOUX APPROACH

Divide the base of the mountain into a grid of 1 meter squares. Measure the altitude of the mountain at the center of each square. The volume on a single grid square is approximately 1 m^2 × (that square's altitude), so the total volume is 1 m^2 times the sum of the altitudes.

The Lebesgue approach

Draw a contour map of the mountain, where adjacent contours are 1 meter of altitude apart. The volume of earth a single contour contains is approximately 1 m × (that contour's area), so the total volume is the sum of these areas times 1 m.

Folland summarizes the difference between the Riemann and Lebesgue approaches thus: "to compute the Riemann integral of *f*, one partitions the domain [*a*, *b*] into subintervals", while in the Lebesgue integral, "one is in effect partitioning the range of *f* ."

We have introduced the notion of a measurable space (X, X) consisting of a set X and a $\sigma-$algebra X of subsets of X. We now consider certain functions which are defined on X and have real, or extended real values. These functions, which will be called "measures," are suggested by our idea of length, area, mass, and so forth. Thus it is natural that they should attach the value 0 to the empty set ϕ and that they should be additive over disjoint sets in X. (Actually we shall require that they be countably additive in the sense to be described below.) It is also desirable to permit the measures to take on the extended real number $+\infty$.

8.4 DEFINITIONS AND PROPERTIES OF MEASURES

8.1 DEFINITION. A measure is an extended real-valued function μ defined on a $\sigma-$algebra X of subsets of X such that $(i)\,\mu(\phi)=0, (ii)\,\mu(E)\geq 0$ for all $E \in X$, and $(iii)\,\mu$ is countably additive in the sense that if (E_n) is any disjoint sequence of sets in X, then

$$(8.1)\ \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Since we permit μ to take on $+\infty$, we remark that the appearance of value $+\infty$ on the right side of the equation (8.1) means either that $\mu(E_n) = +\infty$ for some n or that the series of nonnegative terms on the right side of (8.1) is divergent. If a measure foes not take on $+\infty$,

*This means that $E_n \cap E_m = \phi$ if $n \neq m$.

We say that it is finite. More generally, if there exists a sequence (E_n) of sets in X with $X = \cup E_n$ and such that $\mu(E_n) < +\infty$ for all n, then we say that μ is $\sigma-$finite.

8.2 EXAMPLES. (a) Let X be any nonempty set and let X be the $\sigma-$ algebra of all subsets of X. Let μ_1 be defined on X by

$$\mu_1(E) = 0, \qquad \text{for all } E \in X;$$

And let μ_2 be defined by

$$\mu_2(\phi) = 0, \quad \mu_2(E) = +\infty \qquad \text{if } E \neq \phi.$$

Both μ_1 and μ_2 are measures, although neither one is very interesting. Note that μ_2 is neither finite nor $\sigma-$finite.

(b) Let (X, X) be as in (a) and let p be a fixed element of X. Let μ be defined for $E \in X$ by

$$\begin{aligned} \mu(E) &= 0, \quad \text{if } p \notin E, \\ &= 1, \quad \text{if } P \in E. \end{aligned}$$

It is readily seen that μ is a finite measure; it is called the unit measure concentrated at p.

(c) Let $X = N = \{1,2,3,\ldots\}$ and let X be the $\sigma-$algebra of all subsets of N. If $E \in X$, define $\mu(E)$ to be equal to the number of elements in E if E is a finite set and equal to $+\infty$ if E is an infinite set. Then μ is a measure and is called the counting measure on N. Note that μ is not finite, but it is $\sigma-$finite.

(d) If $X = R$ and $X = B$, the Borel algebra, then it will be shown in Chapter 9 that there exists a unique measure λ defined on B which coincides with length on open intervals. [By this we mean that if E is the nonempty interval (a,b), then $\lambda(E) = b - a$.] This unique measure is usually called Lebesgue (or Borel) measure. It is not a finite measure, but it is $\sigma-$finite.

(e) If $X = R, X = B,$ and f is a continuous monotone increasing function, then it will be shown in Chapter 9 that there exists a unique measure λ_f defined on B such that if $E = (a,b)$, then $\lambda_f(E) = f(b) - f(a)$. This measure λ_f is called the Borel-Stieltjes measure generated by f.

We shall now derive a few simple results that will be needed later.

8.3 LEMMA. Let μ be a measurable defined on a $\sigma-$algebra X. If E and F belong to X and $E \subseteq F$, then $\mu(E) \le \mu(F)$. If $\mu(E) < +\infty$, then $\mu(F/E) = \mu(F) - \mu(E)$.

PROOF. Since $F = E \cup (F/E)$ and $E \cap (F/E) = \phi$, it follows that

$$\mu(F) = \mu(E) + \mu(F/E).$$

Since $\mu(F/E) \ge 0$, it follows that $\mu(F) \ge \mu(E)$. If $\mu(E) < +\infty$, then we can subtract it from both sides of this equation.

8.4 LEMMA. Let μ be a measurable defined on a $\sigma-$algebra X.

(a) If (E_n) is an increasing sequence in X, then

Notes

$$(8.2)\ \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n).$$

(b) If (F_n) is a decreasing sequence in X and if $\mu(F_1) < +\infty$, then

$$(8.3)\ \mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim \mu(F_n).$$

PROOF. (a) If $\mu(E_n) < +\infty$ for some n, then both sides of equation (8.2) are $+\infty$. Hence we can suppose that $\mu(E_n) < +\infty$ for all n.

Let $A_1 = E_1$ and $A_n = E_n / E_{n-1}$ for n > 1. Then (A_n) is a disjoint sequence of sets in X such that

$$E_n = \bigcup_{j=1}^{n} A_j, \quad \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n.$$

Since μ is countably additive,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \lim \sum_{n=1}^{m} \mu(A_n).$$

By Lemma 8.3 $\mu(A_n) = \mu(E_n) - \mu(E_{n-1})$ for n > 1, so the finite series on the right side telescopes and

$$\sum_{n=1}^{m} \mu(A_n) = \mu(E_m).$$

Hence equation (8.2) is proved.

(b) Let $E_n = F_1 / F_n$, so that (E_n) is an increasing sequence of sets in X. If we apply part (a) and Lemma 8.3, we infer that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n) = \lim\left[\mu(F_1) - \mu(F_n)\right]$$
$$= \mu(F_1) - \lim \mu(F_n).$$

Since $\bigcup_{n=1}^{\infty} E_n = F_1 / \bigcup_{n=1}^{\infty} F_n$, it follows that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu(F_1) - \mu\left(\bigcap_{n=1}^{\infty} F_n\right).$$

Combining these two equations, we obtain (8.3)

8.5 DEFINITION. A measure space is a triple (X, X, μ) consisting of a set X, a $\sigma -$algebra X of subsets of X, and a measure μ defined on X.

There is a terminological matter that needs to be mentioned and which shall be employed in the following. We shall say that a certain proposition holds $\mu -$almost everywhere if there exists a subset $N \in X$ with $\mu(N) = 0$ such that the proposition holds on the complement of N. Thus we say that two functions f, g are equal $\mu -$almost everywhere or that they are equal for $\mu -$almost all x in case $f(x) = g(x)$ when $x \notin N$, for some $N \in x$ with $\mu(N) = 0$. In this case we will often write

$$f = g, \ \mu - \text{a.e.}$$

In like manner, we say that a sequence (f_n) of function on X converges $\mu -$almost everywhere (or converges for $\mu -$almost all x) if there exists a set $N \in X$ with $\mu(N) = 0$ such that $f(x) = \lim f_n(x)$ for $x \notin N$. In this case we often write

$$f = \lim f_n, \ \ \mu - \text{a.e.}$$

Of course, if the measure μ is understood, we shall say "almost everywhere" instead of "μ-almost everywhere."

There are some instances (suggested by the notion of electrical charge, for example) in which it is desirable to discuss functions which behave like measures except that they take both positive and negative values. In this case, it is not so convenient to permit the extended real numbers $+\infty, -\infty$ to be values since we wish to avoid expressions of the form $(+\infty) + (-\infty)$. Although it is possible to handle "singed measures" which take on only one of the values $+\infty, -\infty,$ we shall restrict our attention to the case where neither of these symbols is permitted. To indicate this restriction, we shall introduce the term “charge,” which is not entirely standard.

8.6 DEFINITION. If X is a $\sigma-$ algebra of subsets of a set X, then a real-valued function λ defined on X is said to be a charge in case $\lambda(\phi)=0$ and λ is countably additive in the sense that if (E_n) is a disjoint sequence of sets in X, then

$$\lambda\left(\bigcup_{n=1}^{\infty} E_n\right)=\sum_{n=1}^{\infty}\lambda(E_n).$$

[Since the left-hand side is independent of the order and this equality is required for all such sequences, the series on the right-hand side must be unconditionally convergent for all disjoint sequences of measurable sets.]

It is clear that the sum and different of two charges in a charge. More generally, any finite linear combination of charges is a charge. It will be seen in Chapter 5 that functions which are integrable over a measure space (X, X, μ) give rise to charges. Later, in Chapter 8, we will characterize those charges which are generated by integrable functions.

EXERCISES

8.A. If μ is a measure on X and A is a fixed set in X, then the function λ, defined for $E \in X$ by $\lambda(E)=\mu(A \cap E)$, is a measure X.

8.B. If $\mu_1,\ldots,\mu_n$ are measure on X and $a_1,\ldots a_n$ are nonnegative real numbers, then the function λ, defined for $E \in X$ by

$$\lambda(E)=\sum_{j=1}^{n} a_j\mu_j(E),$$

Is a measure on X.

8.C. If (μ_n) is a sequence of measures on X with $\mu_n(X)=1$ and if λ is defined by

$$\lambda(E) = \sum_{n=1}^{\infty} 2^{-n} \mu_n(E),\ E \in X,$$

Then λ is a measure of X and $\lambda(X) = 1$.

8.D. Let $X = N$ and let X be the $\sigma -$algebra of all subsets of N. If (a_n) is a sequence of nonnegative real numbers and if we define μ by

$$\mu(\phi) = 0;\ \mu(E) = \sum_{n \in E} a_n,\ \ E \neq \phi;$$

Then μ is a measure on X. Conversely, every measure on X is obtained in this way for some sequence (a_n) in $\bar{R}^+$.

8.E. Let X be an uncountable set and let X be the family of all subsets of X. Define μ on E in X by requiring that $\mu(E) = 0,$ if E is countable, and $\mu(E) = +\infty,$ if E is uncountable. Show that μ is a measure on X.

8.F. Let $X = N$ and let X be the family of all subsets of N. If E is finite, let $\mu(E) = 0;$ if E is infinite, let $\mu(E) = +\infty.$ Is μ measure on X?

8.G. If X and X are as in Exercise 8.F, let $\lambda(E) = +\infty$ for all $E \in X.$ Is λ is a measure?

8.H. Show that Lemma 8.4(b) may fail if the finiteness condition $\mu(F_1) < +\infty$ is dropped.

8.1. Let (X, X, μ) be a measure space and let (E_n) be a sequence in X. Show that

$$\mu(\lim\inf E_n) \leq \lim\inf \mu(E_n).$$

[See Exercise 2.E]

8.1. Using the notation of Exercise 2.D, show that

$$\lim\sup \mu(E_n) \leq \mu(\lim\sup E_n)$$

Where $\mu(\cup E_n) < +\infty$. Show that this inequality may fail if $\mu(\cup E_n) + \infty$.

8.K. Let (X, X, μ) be a measure space and let $Z = \{E \in X : \mu(E) = 0\}$. Is Z a $\sigma -$algebra? Show that if $E \in Z$ and $F \in X$, then $E \cap F \in Z$. Also, if E_n belongs to Z for $n \in N$, then $\cup E_n \in Z$.

8.L. Let X, X, μ, Z be as in Exercise 8.K and let $X^{'}$ be the family of all subsets of X of the form

$$(E \cup Z_1) / Z_2,\ E \in X,$$

Where Z_1 and Z_2 are arbitrary subsets belonging to Z. Show that asset is in $X^{'}$ if and only if it has the form $E \cup Z$ where $E \in X$ and Z is a subset of a set in Z. Show that the collection $X^{'}$ forms a $\sigma -$algebra of sets in X. The $\sigma -$algebra $X^{'}$ is called the completion of X (with respect to μ).

8.M. With the notation of Exercise 8.L, let $\mu^{'}$ be defined on $X^{'}$ by

$$\mu^{'}(E \cup Z) = \mu(E),$$

When $E \in X$ and Z is a subset of a set in Z. Show that $\mu^{'}$ is well-defined and is a measure $X^{'}$ which agrees with μ on X. The measure $\mu^{'}$ is called the completion of μ.

8.N. Let (X, X, μ) be a measure space and let $(X, X^{'}, \mu^{'})$ be its completion in the sense of Exercise 8.M. Suppose that f is an $X^{'} -$ measurable function on X to $\bar{R}$. Show that there exists an X-measurable function g on X to $\bar{R}$. which is $\mu -$almost everywhere equal of f. (Hint: For each rational number r, let $A_r = \{x : f(x) > r\}$ and write $A_r = E_r \cup Z_r$, where $E_r \in X$ and Z_r is a subset of a set in Z. Let Z be a set in Z containing $\cup Z_r$ and define $g(x) = f(x)$ for $x \notin Z$, and

$g(x) = 0$ for $x \in Z$. To show that g is X-measurable, use Exercise 2. $\cup$.)

8.O. Show that Lemma 8.4 holds if μ is a charge on X.

8.P. If μ is a charge on X, let π be defined for $E \in X$ by

$$\pi(E) = \sup\{\mu(A) : A \subseteq E, A \in X\}.$$

Show that π is a measure on X. (Hint: If $\pi(E_\pi) < \infty$ and $\in > 0$, let $F_n \in X$ be such that $F_n \subseteq E_\pi$ and $\pi(E_\pi) \leq \mu(F_n) + 2^{-n} \in$.)

8.Q. If μ is a charge on X, let υ be defined for $E \in X$ by

$$\upsilon(E) = \sup \sum_{j=1}^{n} \left|\mu(A_j)\right|,$$

Where the supremum is taken over all the finite disjoint collections $\{A_j\}$ in X with $E = \bigcup_{j=1}^{n} A_j$. Show that υ is a measure on X. (It is called the variation of μ.)

8.R. Let λ denote Lebesgue measure defined on the Borel algebra B of R [See Example 8.2(d)]. (a) If E consists of a single point, then $E \in B$ and $\lambda(E) = 0$. (b) If E is countable, then $E \in B$ and $\lambda(E) = 0$.

(c) The open interval (a, b), the half-open intervals (a, b], [a, b), and the closed interval [a, b] all have Lebesgue measure b –a.

8.S. If λ denotes Lebesgue measure E is an open subset of R, then $\lambda(E) > 0$ if and only if E is nonvoid. Show that if K is a compact subset of R, then $\lambda(K) < +\infty$.

8.T. Show that the Lebesgue measure of the Cantor set is zero.

8.U. By varying the construction of the Cantor set, obtain a set of positive Lebesgue measure which contains no nonvoid open interval.

Notes

8.V. Suppose that E is a subset of a set $N \in X$ with $\mu(N) = 0$ but that $E \notin X$. The sequence $(f_n), f_n = 0,$ converges $\mu -$almost everywhere to X_E. Hence the almost everywhere limit of a sequence of measurable functions may not be measurable.

Check your progress

1. Prove: Let μ be a measurable defined on a $\sigma -$algebra X. If E and F belong to X and $E \subseteq F,$ then $\mu(E) \le \mu(F)$. If $\mu(E) < +\infty,$ then $\mu(F / E) = \mu(F) - \mu(E)$.

___

___

___

2. Prove: Let μ be a measurable defined on a $\sigma -$algebra X.

(a) If (E_n) is an increasing sequence in X, then

$$(8.2) \quad \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n).$$

(b) If (F_n) is a decreasing sequence in X and if $\mu(F_1) < +\infty,$ then

$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim \mu(F_n).$$

___

___

___

8.5 LET US SUM UP

1. A measure is an extended real-valued function μ defined on a $\sigma -$ algebra X of subsets of X such that $(i)\mu(\phi) = 0, (ii)\mu(E) \ge 0$ for all

$E \in X$, and $(iii)\,\mu$ is countably additive in the sense that if (E_n) is any disjoint sequence of sets in X, then

$$(8.1)\ \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

2. Let μ be a measurable defined on a $\sigma -$algebra X. If E and F belong to X and $E \subseteq F$, then $\mu(E) \le \mu(F)$. If $\mu(E) < +\infty$, then $\mu(F/E) = \mu(F) - \mu(E)$.

3. Let μ be a measurable defined on a $\sigma -$algebra X.

(a) If (E_n) is an increasing sequence in X, then

$$(8.2)\ \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n).$$

(b) If (F_n) is a decreasing sequence in X and if $\mu(F_1) < +\infty$, then

$$(8.3)\ \mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim \mu(F_n).$$

4. If X is a $\sigma -$ algebra of subsets of a set X, then a real-valued function λ defined on X is said to be a charge in case $\lambda(\phi) = 0$ and λ is countably additive in the sense that if (E_n) is a disjoint sequence of sets in X, then

$$\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n).$$

8.6 KEY WORDS

Lebesgue Integration

Measures

Riemann- Darboux approach

Real valued function

8.7 QUESTIONS FOR REVIEW

1. Explain about Lebesgue integration

2. Prove: Let μ be a measurable defined on a $\sigma-$algebra X.

(a) If (E_n) is an increasing sequence in X, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n).$$

(b) If (F_n) is a decreasing sequence in X and if $\mu(F_1) < +\infty,$ then

$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim \mu(F_n).$$

8.8 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle

2. The elements of Integration and Lebesgue Meassure

3. A course on integration- Nicolas Lerner

4. General theory of Integration- Dr. E.W. Hobson

5. General theory of Integration- P.Muldowney

6. General theory of functions and Integration- Angus E.Taylor

8.9 ANSWERS TO CHECK YOUR PROGRESS

1. See section 8.4

2. See section 8.4

3. See section 8.4

UNIT-9 MODES OF CONVERGENCE

STRUCTURE

9.0 OBJECTIVE

In this unit we will learn and understa22nd about Definitions, Egorow's theorem, Luzin's theorem, Riesz sub sequence theorems and their proofs.

9.1 INTRODUCTION

In this section we will study a number of modes of convergence that are of importance in analysis and in the theory of probability. We will examine to some detail the relations between these types of convergence. We finish with a pair of necessary and sufficient conditions for a sequence of functions in C(I) to be convergent in mean. Throughout this section we will suppose that I is a compact interval in R. While most of the results presented here have extensions to unbounded intervals (see Section 20), some additional hypotheses may be needed in that case.

Almost Uniform Convergence

We have already discussed uniform convergence, point wise convergence and a.e. convergence. We now introduce another mode of convergence of functions in M(I) that is often useful. Intuitively, almost uniform convergence of a suqence in M(I) means that, outside of certain subsets of I having arbitrarily small measure, one has uniform convergence.

WARNINGS. (a) This is not the same thing as saying that one has uniform convergence outside of a null set. See Exercises 9.C and 9.D.

(b) This use of the word "almost" is in slight conflict with the 'almost everywhere" terminology. However, we will use it because it is quite standard.

- **9.1 Definition.** (a) A section $(f_n)\, in\, M(I)$ is said to be almost uniformly convergent to a function $f \; on\, I; = [a,b]\, if \;\; for\, every\, \gamma > 0$ there exists a measurable set $E_\gamma \subseteq I \; with \left|E_\gamma\right| \leq \gamma \, such\, that\, (f_n)$ converges to I uniformly on the set $I = E_\gamma$ in this case we sometimes write $f_n \to f[a,u]\, on\, I.$

(b) We say that a sequence $(f_n)\, in\, M(I)$ is almost uniformly Cauchy on I if for every $\gamma > 0$ there exists a measurable set $E_\gamma \subseteq I \; with \left|E_\gamma\right| \leq \gamma$ such that (f_n) is a uniform Cauchy sequence on $I - E_\gamma$. [this means that for every $\in > 0$ there exists $N(\in) \in N \; such\, that\, if \;\; m > n \geq N(e)$ and $x \in I - E_\gamma, then |\; f_m(x) - f_n(x) | \leq \in .]$

It is an easy exercise to show that if a sequence $(f_n) in\, M(I)$ is almost uniformly convergent to $f \; on\, I$, then it is almost uniformly Cauchy on I. Morever, this sequence converges a.e. to $f \; on\, I$ so that $f \in M(I)$.We now establish a result in the converse direction.

- **9.2 Lemma**. If a sequence $(f_n)\, inM(I)$ is almost uniformly Cauchy on I, then there exists a functions $f \notin M(I)$ such that (f_n) converges almost uniformly and almost everywhere to f.

Proof. If $k \in N, let\, E_k \in M(I)$ be such that $|E_k| \le 1/2^k\, and\, (f_n)$ is uniformly Cauchy and therefore uniformly convergent on $I - E_k$. Let $F :- \lim\sup_k E_k$, we have $|F| = 0$. If follows from the definition of F that if $x \in I - F$, then there exists k_x such that $x \in I - E_k\, for\, all\, k \ge k_x$.

Therefore,

$\lim_n f_n(x)\, exists\ for\, all\, x \in I - F\ and\ we\, define\ f\ on\, I\, by\ f(x) := \lim_n f_n(x)\, for\, x \in I - Fand\ f(x); = 0$ for $x \in F$. Therefore $F_n \to fa.e.and\ f \in M(I)$.

To see that the convergence is almost uniform, let $\gamma > 0$ be given and let K be such that $1/2^{K-1} \le \gamma. If\ F_K := \bigcup_{j=K}^{\infty} E_j$, then it follows from

$(10\varsigma)\, that\quad |F_K| \le \sum_{j=K}^{\infty} |E_j| \le 1/2^{K-1} \le \gamma.$

Since $I - F_K \subseteq I - E_K$, the sequence (f_n) is a uniform Cauchy sequence on $I - F_K$, whence it follows that (f_n) converges to f uniformly on $I - F_K$ Q.E.D.

We now establish and important theorem, proved in 1911 by the Russian mathematician Dmitrii Fedorovich Egorow (=D.Th.Egorof). As this result is stated here, it is valid only for compact intervals. In Section 20 we will give a formulation of this result for unbounded intervals.

9.2 EGOROW'S THEOREM.

Let I be a compact interval and let (f_n) be a sequence in M(I) that converges almost everywhere to $f \in M(I)\, on\, I$. Then the sequence (f_n) converges almost uniformly to $f\ on\, I$.

Proof. We suppose without loss of generality that (f_n) converges to f at every point of I. If $m, n \in N$, we let

$$E_n(m) := \bigcup_{k=n}^{\infty} \{|f_k - f| \ge 1/m\},$$

So that $E_n(m) \in M(I)\, and\, E_{n+1}(m) \subseteq E_n(m). Since\ f_n \to f\ on\, I, then$

$$\bigcap_{n=1}^{\infty} E_n(m) = \theta.$$

Therefore $(10, \delta)$ implites that

$|E_n(m)| \to 0\, as\, n \to \infty,\ for\, each\, m \in N. If\ \gamma > 0$ is given, for each

$m \in N$, $we\,let\,k_m \in N$ be such that $\left|E_{k_m}(m)\right| \leq \gamma / 2^m$ and set

$E_\gamma := \bigcup_{m=1}^{\infty} E_{k_m}(m)$. Therefore $E_\gamma \in M(I)\,and\left|E_\gamma\right| \leq \gamma\,by(10.\varsigma)$. We note that if $x \notin E_\gamma, then\, x \notin E_{k_m}(m)$ for every $m \in N$, so that

$\left| f_k(x) - f(x)\right| < 1/m$

For all $k \geq k_m$. Therefore - (f_n) is uniformly convergent to

$f\,on\,I - E_\gamma .Q.E.D.$

As an application of Egorov's theorem, we will establish one form of a remarkable therorem, proved in 1912 by Nikolal Nikolaevich Luzin (=Lusin) (1883-1950).

9.3 LUZIN'S THEOREM

If f belongs to $M(I)$. Then given $\gamma > 0$ there exists a measurable set $E_\gamma \subseteq I\, with\left|E_\gamma\right| \leq \gamma$ such that the restriction of $f\, to\, F_\gamma := I - E_\gamma$ is continuous on F_γ.

- **Proof.** Since $f \in M(I)$, it follows from Theorem 6.7 that thee exists a sequence (h_k) of continuous functions that converges to $f\, a.e. on\, I$. In view of Egorov's Theorem, for each $\gamma > 0$ there exists a set $E_\gamma \in M(I)\, with\left|E_\gamma\right| \leq \gamma$ such that (h_n) converges to f uniformly on $F_\gamma := I - E_\gamma$. Of course, the restriction $h_n f \mid F_\gamma$ is continuous on F_γ and since this sequence converges to $f \mid F_\gamma$ uniformly on F_γ we conclude that the restriction $f \mid F_\gamma$ is continuous. Q.E.D.
- **Remarks** (a) One should not misunderstand the assertion in Luzin's Thereorem. It is not begin claimed that f is continuous at any point of Z. Note that if f is Dirichlet's function from Example 2.3(a), then there is a null set Z such that the restriction of to $[0,1] - Z$ is continuous; however, f is not continuous at any point of $[0,1]$.

(b) Another form of Luzin's Theorem is that if $f \in M(I)$, then for every $\gamma > 0$, there exists a continuous function g on I such that $|\{f \neq g\}| \leq \gamma$. (See Theorem 20.18.)

Convergence in Measure

There is another mode of convergence for measurable functions that is particularly important in probability theory. First we note that if $f_n, f \in M(I)$. then (by Theorem 6.1) the function $|f_n - f|$ is also measurable and therefore (by Theorem 10.5) the set $\{|f_n - f| \geq \alpha\}$ is a measurable set in I.

Definition. (a) A sequence $(f_n) \in M(I)$ converges in measure (or converges in probability) to $f \in M(I)$ if for every $\alpha > 0,$ we have

$$(11.\alpha) \quad \lim_{n\to\infty} \left|\{|f_n - f| \geq \alpha\}\right| = 0.$$

In this case we sometimes write

$f_n \to f\ [meas] on I.$

(b) A sequence $(f_n) \in M(I)$ is Cauchy in measure if for every $\alpha > 0.$ we have

$$(11,\beta) \quad \lim_{m,n\to\infty} \left|\{|f_m - f_n| \geq \alpha\}\right| = 0.$$

It seems reasonable, but is not obvious, that a sequence that converges in measure is also Cauchy in measure, and that the limit of a sequence that converges in measure is unique a.e. We now state these results formally.

- 9.6 Lemma. (a) If (f_n) converges in measure to $f, then\left(f_n\right)$ is Cauchy in measure.

(b)If $\left(f_n\right)$ converges in measure to f and also to g then $f = a.e.$

Proof. (a) It follows from the Triangle Inequality that

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)|,$$

Whence we infer that

$$\{|f_m - f_n| \geq \alpha\} \subseteq \left\{|f_m - f| \geq \frac{1}{2}\alpha\right\} \cup \left\{|f - f_n| \geq \frac{1}{2}\alpha\right\}.$$

Since the measures of the sets on the right side approach 0 as $m, n \to \infty,$ the statement follows from $(11,\alpha)$ and the fact that $|A \cup B| \leq |A| + |B|.$

The proof of (b) is similar and is an exercise.

We need to relate the notions of convergence in measure $[=meas]$ with those of almost uniform $[=a,u.]$ convergence and convergence in mean $[=meas]$.

- **9.7 Lemma**. (a) If a sequence $(f_n)\in M(I)$ converges almost uniformly to $f\in M(I)\, on\, I,$ then it converges in measure to $f\ on\, I.$

(b) If a sequence $(f_n)\in R^*(I)$ then it converges in measure to $f\ on\, I.$

Proof. (a) Let $\alpha>0$ be given By hypothesis, for every $m\in N$ there exists $E_m\in M(I)$ with $|E_m|\le 1/m$ such that (f_n) converges uniformly to f on $I-E_m$. Consequently, there exists $N(\alpha,m)\in N$ such that if $n\ge N(\alpha,m)$ then the set $\{|f_n-f|\ge\alpha\}\subseteq$ so that

$$|\{|f_n-f|\ge a\}|\le|E_m|\le 1/m.$$

Since m is arbitrary, we conclude that $f_n\to f$ [meas] on I .

(b) Let $a>0$ be given and let $F_n=(|f_n-f|\ge\alpha)\in M(I).$ Since we have $\alpha.1_{F_n}\le|f_n-f|,$ it follows from Corollary 3.3 or from Chebysber's Inequality that

$$\alpha|F_n|\le\int_I|f_n-f|=\|f_n-f\|.$$

But since $\|f_n-f\|\to 0,$ we conclude that $|F_n|\to 0\, as\, n\to\infty$ for each fixed $\alpha>0. Thus\ f_n\to f$ [mean] on $I.$ Q.E.D.

The next examples show that there are some drastic difference between convergence in measure and the other modern of convergence.

9.8 Examples. (a) let $h_n:=n.1_{(0.1/n)}$ on the internal $I:=[0.1]\ for\ n\in N.$ It is an exercise to show that the sequence $[h_n]$ converges everywhere (and hence almost everywhere),almost uniformly, and in measure to the zero function on $I.$However, this sequence does not converge in mean.

(b) Let (f_K) be the sequence of functions defined in Exercise 8.F. It was shown that this sequence converges in mean to the zero function. Therefore, it follows from Theorem 9.7(b) that it converges in measure. However, it was also seen in that exercise that this sequence does not converge at any point of I, so it does ot converge a.e. or a.u on I.

It was seen in the proof of the Completenses Theorem 9.12 that a sequence that a sequence that in Cauchy in mean but a subsequence that is e.e. convergent. We will now show that a sequence that a Cauchy is measure has a subseqeucne that is a.e. convergent. (This result is due to F. Riest.)

9.4 RIESZ SUBSEQUENCE THEOREM.

If $(f_n) \in M(I)$ is Cauchy in measure, then there exist a subsequence $(f_{n_n}) f \in M(I)\, such\, that\; f_{n_n} \to f$ almost everywhere, almost uniformly and in measure on $I = (a,b)$

In face the entire sequence (f_n) converges in measure to f .

Proof. If (f_n) is Cauchy in measure, it is an exercise to show that for every $\alpha > 0$ there exists $N(\alpha) \in N$ such that if $m > n \geq N(\alpha)$, then.

$\left|\left\{|f_m - f_n| \geq \alpha\right\}\right| \leq \alpha.$

We let $n_1 := N(1/2)$ and inductively define $n_{k+1} = \max\left\{n_k + 1, N(1/2^k)\right\}.$

Now set $g_k := f_{n_k}$ to obtain a subsequence of (f_n) with the property that if $E_k := \left\{|g_{k+1} - g_k| \geq 1/2^k\right\}, then |E_K| \leq 1/2^k.$

$(11.\gamma) \quad \leq 1/2^{k-1} + ... + 1/2^i$

$\leq 1/2^{i-1}.$

Therefore it follows that $(g_k(x))$ converges for each $x \in I - F.$ WE define $f(x) := \lim_k g_k(x)\, for\, x \in I - F and\; f(x) := 0\, for\, x \in F.$

Consequently $g_k \to f[a,e]\, on\, I\, and\; f \in M(I).$

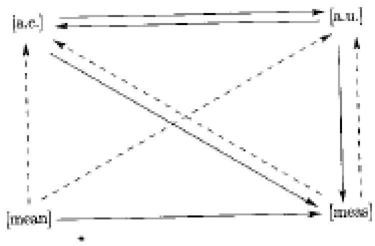
It remains to show that the original sequence (f_n) converges in measure to $f\, on\, I.$ Indeed, an argument similar to that in Lemma 9.6 shows that

$$\left|\left\{|f - f_n| \geq \alpha\right\}\right| \leq \left|\left\{|f - f_{n_k}| \geq \frac{1}{2}\alpha\right\}\right| + \left|\left\{|f_{n_k} - f_n| \geq \frac{1}{2}\alpha\right\}\right|.$$

Now the first term on the right approaches 0 since the subsequence (f_{n_k}) converges in measure to f, and the second term approaches 0 since the sequence (f_n) is Cauchy in measure. Q.E.D.

It is useful to summarize the implications that have been established concerning the various modes of convergence. Following M.E. Munroe, we do so in the following diagram. Here a solid arrow signifies implication, a dashed arrow signifies that a subsequence converges in the indicated mode, and the absence of an arrow indicates that a counterexample can be established. (Thus Egorov's Theorem is indicated by the solid arrow from [meas]to [a.u]. Similarly, convergence is measure does not imply convergence in mean, in general.)

The reader should verify the implications that are indicated in this diagram, and show that no other implications are valid without additional hypotheses.



* Diagram 11.1 Compact interval.

Mean Convergence We have seen from diagram 9.1 that none of the solid arrows ends at [mean].Therefore, we will insert several results here tat have mean convergence as a conclusion. The first one is a version of the Mean Convergence. Theorem 8.9 for a sequence that is Catchy in measure and satisfies a domination condition.

- **Dominated Convergence Theorem**. Suppose that $(f_n) \subset R^*(I)$ is Cauchy in measure on $I \; and \; \alpha, \omega \in R^*(I)$ are such that for each $n \in N.$

(11.δ) $\alpha(x) \le f_n(x) \le \omega(x) \quad for \; a.e. \; x \in I.$

Then there exists $f \in R^*(I) \; such \, that \|f - f_n\| \to 0.$

Proof. By the Riesz Subsequence Theorem 9.9, there exists $f \in M(I)$ such that (f_n) converges to f is measure. If (f_n) does not converge in mean to f ,there exists $\in_0 > 0$ and a subsequence

$(h_K) \, of \, (f_n) \, such \, that \|h_k - f\| \ge \in_o$. Since (f_n) converges in measure to f , so does its subsequence (h_k). that converges a.e. to f. By the Mean Convergence Theorem 8.9, the subsequence $(h_{k(r)})$ converges is mean to f.which contradicts that $\|h_k - f\| \ge \in_0$ for all $k \in N.$

Our next result shows that the mean convergence of a sequence $(f_n) in R^*(I)$ in a functions f takes place when the sequence (f_n) converges in measure to f and to Cauchy in mean.

- **Theorem.** Suppose that $(f_n) \subset R^*(I)$ is Cauchy in mean and converges in measure to a function f. Then $f \in R^*(I) and \| f - f_n \| \to 0.$

Proof. Since $\| f_m - f_n \| \to 0 \, as \, m, n \to \infty, given \in > 0,$ N_g such that if $m > n \geq N_{g,}$ then

(9.g) $$\int_I | f_m - f_n | \leq \in .$$

Since $f_n \to f$ on $I,$ it follows from the Riesz Subsequence. Theorem. 9.9 that there exist a subsequence $(g_k) of \, (f_n) \, and \, g \in M(I),$ such that $g_k \to g[a.e.]$ and [meas] on I. Since (g_k) is a subsequence of $(f_n),$ Lemma 9.6(b) implies that $g = f$ a.e. Now replace $f_m in (11. \in) by \, g_k$ for k sufficiently large and apply Faton's Lemma 8.7 to conclude that

$$\int_I | f - f_n | \leq \liminf_{k \to \infty} \int_I |g_k - f_n| \leq \in$$

For all $n \geq N_{\in}$ .Since $\in > 0$ is arbitrary, we have $\| f - f_n \| \to 0.$ Q.E.D

The Vitali Convergence Theorems

We conclude this section with two theorems that characterize mean convergence for a sequence in $L(I)$. First, it will be seen that if $f_n \to f$ [a.e.] on $I, then \, f_n \to f \, [mean]$ if and only if the mapping $E \mapsto \| f_n \| E$ defined in equation $(10.\lambda)$ satisfies either of the uniformity conditions that we now define.

Definition. (a) A collection $F \subset L(I)$ is uniformly absolutely continuous if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $f \in F$ and $E \in M(I), |E| \leq \delta_\varepsilon, then$

$$\| f \| E := \int_E | f | \leq \varepsilon.$$

(b) A collection $F \subset L(I)$ is uniformly integrable if for every $\varepsilon > 0$ there exists $K = K_\varepsilon \in N$ such that if $f \in F \, and \, H_{f,K} := \{ | f | \geq K \},$ then

$$\|f\|H_{f,K} := \int_{H_{I,K}} |f| \leq \varepsilon.$$

Intuitively, uniform absolute continuity requires that the integrals $f_g|f|, f \in F$, are uniformly small when $|E|$ is small. Similarly, uniform integrability requires that the integrals of $f \in F$ over the sets where $|f|$ is large are uniformly small.

Vitali Convergence Theorem. I. Let $I := [a,b]$ be a compact interval and let (f_n) be a sequence in $L(I) with\, f_n \to f[a.e.] on\, I$.

Then the following statements are equivalent.

(a) $f \in L(I)\, and \|f_n - f\| \to 0\, as\, n \to \infty.$

(b) The set $\{f_n : n \in N\}$ is uniformly absolutely continuous.

(c) The set $\{f_n : n \in N\}$ is uniformly integrable.

Proof. $(a) \Rightarrow (b)$ Given $\in > 0$ there exists n_1 such that if $n > n_1$, then $|f_n|E - \|f\|\varepsilon \leq \|1_E f_n 1_E\| \leq \|f_n - f\| \leq \frac{1}{2}\varepsilon$, whence it follows that

$$|f_n|E - \|f\|E \leq \frac{1}{2}\varepsilon + \|f\|_E \; for\, n > n_1, E \in M(I).$$

$(b) \Rightarrow (a)$ Given $\varepsilon > 0$ let $\delta_\varepsilon > 0$ be such that if $|E| \leq \delta_{\varepsilon,} n \in N, then \|f_n\|E \leq \varepsilon. Since |f_n| \to |f|[a.e.]$ on I and hence on E, Fatou's Lemma 8.7 implies that $f \in L(I)\, and$

$$\|f\|_E \leq \liminf_{n\to\infty} \|f_n\|_E \leq \varepsilon.$$

Egorov's theorem 9.3 implies that there exists a set $B \in M(I)\, with |B| \leq \delta_\varepsilon$ such that $f_n \to f$ uniformly on $I - B$. Therefore

$$\|f_n - f\| \leq \|f_n - f\| I - B + \|f_n\| B + \|f\|_B$$

$$\leq \|f_n - f\| I - B + 2\varepsilon.$$

Further, there exists n_o such that if $n \geq n_0 and\, x \in I - B, then |f_n(x) - f(x)| \leq \varepsilon / |I|$, whence we conclude that

$$\| f_n - f \| I - B \le (\varepsilon / |I|).|I - B| \le \varepsilon.$$

Consequently we have $\| f_n - f \| \le 3\varepsilon$ whenever $n \ge n_0$. Since $\varepsilon > 0$ is arbitrary, assertion (a) follows.

$(b) \Rightarrow (c)$ Given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $|E| \le \delta_\varepsilon$ and $n \in N$, then $\| f_n \|_E \le \varepsilon. Let \{I_1, ..., I_M\}$ be a partition of I into nonoverlapping intervals with length $\le \delta_1$.then

$$\int_a^b \left| f_n = \sum_{j=1}^{M} \int_{I_j} |f_n| \le \sum_{j=1}^{M} 1 = M \right|$$

For $n \in N.$ Now let $K = K_g > M / \delta_{g.}$ If $H_{n,K} := \{|f_n| \ge K\}.$ then since $K.1_{Hnk} \le |f_n|$, it follows from the inequality

$$K|H_{n,K}| \le \| f_n \| \le M$$

That $|H_{n,K}| \le M / K \le \delta_{\varepsilon,}$ whence (b) implies that

$$|f_n| H_{n,k} \le \varepsilon \; for \; n \in N.$$

(c) $(c) \Rightarrow (b)$ $Given\, \varepsilon > 0$ let K be such that

$\| f_n \| H_{n,K} \le \frac{1}{2}\varepsilon \; for \; all \; n \in N.$ Now let

$\delta_\varepsilon := \varepsilon / 2K, so\, that\, if\, |E| \le \delta_\varepsilon \; and\; n \in N.$then

$$\| f_n \|_E = \| f_n \|_{E \cap H_{n,k}} + \| f_n \|_E - H_{n,K}$$

$$\le \| fn \| H_{n,K} + K|E| \le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Thus (b) is proved.

We now obtain a version of the Vitali theorem replacing the hypothesis that $f_n \to f$ [a.e.] on I by the hypothesis that $f_n \to f$ [meas] on I.

Vitali Convergence Theorem. II. Let $I := [a,b]$ be a compact interval and let (f_n) be a sequence in $L(I)\, with\, f_n \to f$ in measure on I.

Then the statements (a),(b),and (c) in 9.13 are equivalent.

Notes

Proof. The proofs of $(a)\Rightarrow(b),(b)\Rightarrow(c), and (c)\Rightarrow(b)$ make no reference to the convergence of the sequence and do not need any change .However, the proof of $(b)\Rightarrow(a)$ given above uses a.e. convergence and needs to be modified.

If the sequence (f_n) does not converge in mean to f, there exists $\in_0 > 0$ and a subsequence $(f_n^{'})$ such that $\left\| f_{n^{'}} - f^{'} \right\| \geq \varepsilon_0 > 0$ for all $n^{'}$.

Since the subsequence $(f_n^{'})$ converges in measure to f, the Riesz Subsequence Theorem implies that it has a further subsequence $(f_{n^{''}})$ that converges a.e. to f. Hence, by the Vitali Convergence Theorem I, the sequence $(f_{n^{''}})$ converges in mean to f, which contradicts the above inequality Q.E.D

Exercises

9.A. Show that the following sequences do not converge uniformly on the indicated intervals, but that they do converge a.e. and hence a.u. Given $\gamma > 0$, find a set E_γ with $|E_\gamma| \leq \gamma$ such that the convergence is uniform on $I - E_\gamma$.

(a) $f_n(x) := nx/1+nx) on [0.1].$

(b) $h_n(x) := 1/(1+x^n) on [0.1].$

(c) $\varphi_n(x) := 1/(1+x^n) on [0.2].$

9.B. Same as in Exercise 9.A. Here the functions are 0 at $x = 0.$

(a) $f_n(x) := 1/\sqrt{x}(1+x^n)$ *for* $x \in [0.1]$

(b) $g_n(x) := 1/\sqrt{x}(2+x^n)$ *for* $x \in [0.1]$

9.C. If $n \geq 2, let\ f_n := 1_{(1/n,2/n)} and\ f := 0$ on the interval $I := [0.1].$

(a) $f_n(x) := 1/\sqrt{x}(2-x^n)$ *for* $x \in [0.1].$

9.C If $n \geq 2, let\ f_n := 1_{(1/n,2/n)} and\ f := 0$ on the interval $I := [0.1].$

(a) Show that $f_n \to f$ everywhere (and therefore a.e.) on I, in mean and in measure.

(b) If $\gamma > 0$ is given, show that $f_n \to f$ uniformly on $[\gamma,1]$, so that $f_n \to f$ almost uniformly.

(C) Show that there does not exists a null set Z such that $f_n \to f$ uniformly on $I - Z$.

(d) Show that $\| f_n - f \| 2 \to 0$.

9.D Let $g_n := \sqrt{n} f_n = \sqrt{n}\, on [1/n, 2/n]\, and := 0$ elsewhere, and let g:=0.

(a) Show that $g_n \to g$ everywhere and therefore a.e.) on I, in mean and in measure.

(b) If $\gamma > 0$ is given, show that $g_n \to g$ uniformly on $[\gamma, 1]$, so that $g_n \to g$ almost uniformly.

(c) Show that there does not exist a null set Z such that $g_n \to g$ uniformly on $I - Z$.

(d) Show that $\| g_n - g \| 2 \not\to 0$.

9.E Let $E_n \in M([a,b])\, for\, n \in N$.

(a) show that (1_{En}) converges to 0 uniformly on [a,b] if and only if $E_n = \theta$ for sufficiently large n.

(b) Show that (1_{En}) converges to 0 almost everywhere on [a,b] if and only if lim $\lim sup_{n\to\infty} E_n$ is a null set.

(c) Show that (1_{E_n}) converges to 0 almost everywhere on [a,b] if and only if $\lim sup_{n\to\infty} E_n$ is a null set.

(d) When does (1_{E_n}) converge to 0 almost uniformly?

(e) Show that (1_{E_n}) converges tot 0 in measure on [a,b] if and only if $\lim_{n\to\infty} E_n = 0.$

(f) When does (1_{E_n}) converge to 0 in mean ?

9.F (a) If (f_n) in $M([a,b])$ converges to f in measure, show that any subsequence also converges to f in measure.

Check Your Progress

1. Prove: If a sequence $(f_n) in M(I)$ is almost uniformly Cauchy on I, then there exists a functions $f \notin M(I)$ such that (f_n) converges almost uniformly and almost everywhere to f.

Notes

2. Prove: Let I be a compact interval and let (f_n) be a sequence in M(I) that converges almost everywhere to $f \in M(I)\, on\, I$. Then the sequence (f_n) converges almost uniformly to $f\ on\, I$.

3. Prove: If f belongs to $M(I)$. Then given $\gamma > 0$ there exists a measurable set $E_\gamma \subseteq I\ with |E_\gamma| \le \gamma$ such that the restriction of $f\ to\, F_\gamma := I - E_\gamma$ is continuous on F_γ.

4. Prove: (a) If a sequence $(f_n) \in M(I)$ converges almost uniformly to $f \in M(I)\, on\, I$, then it converges in measure to $f\ on\, I$.

(b) If a sequence $(f_n) \in R^*(I)$ then it converges in measure to $f\ on\, I$.

5. Prove: Riesz Subsequence theorem.

9.5 LET US SUM UP

1. A section $(f_n)\, in\, M(I)$ is said to be almost uniformly convergent to a function $f\ on\, I; = [a,b]\, if\ \ for\, every\, \gamma > 0$ there exists a measurable set

$E_\gamma \subseteq I\ with |E_\gamma| \leq \gamma\ such\,that\ (f_n)$ converges to I uniformly on the set

$I = E_\gamma$ in this case we sometimes write $f_n \to f[a,u]\,on\,I.$

2. If a sequence $(f_n)\,in M(I)$ is almost uniformly Cauchy on I, then there exists a functions $f \notin M(I)$ such that (f_n) converges almost uniformly and almost everywhere to f.

3. Let I be a compact interval and let (f_n) be a sequence in M(I) that converges almost everywhere to $f \in M(I)\,on\,I.$ Then the sequence (f_n) converges almost uniformly to $f\ on\,I.$

4. If f belongs to $M(I)$. Then given $\gamma > 0$ there exists a measurable set $E_\gamma \subseteq I\ with |E_\gamma| \leq \gamma$ such that the restriction of $f\ to\,F_\gamma := I - E_\gamma$ is continuous on F_γ .

5. (a) If a sequence $(f_n) \in M(I)$ converges almost uniformly to $f \in M(I)\,on\,I,$ then it converges in measure to $f\ on\,I.$

(b) If a sequence $(f_n) \in R^*(I)$ then it converges in measure to $f\ on\,I.$

6. If $(f_n) \in M(I)$ is Cauchy in measure, then there exist a subsequence $(f_{n_n})\,f \in M(I)\,such\,that\ f_{n_n} \to f$ almost everywhere, almost uniformly and in measure on $I = (a,b)$ In face the entire sequence (f_n) converges in measure to f .

I. Let $I := [a,b]$ be a compact interval and let (f_n) be a sequence in $L(I) with\ f_n \to f[a.e.]\,on\,I.$

Then the following statements are equivalent

$f \in L(I)\,and\,\|f_n - f\| \to 0\,as\,n \to \infty.$ The set $\{f_n : n \in N\}$ is uniformly absolutely continuous. The set $\{f_n : n \in N\}$ is uniformly integrable.

9.6 KEY WORDS

convergent to a function

converges in measure to f

Uniformly absolutely continuous

9.7 QUESTIONS FOR REVIEW

1. Explain about Egorow’s theorem.

2. Prove: Luzin’s theorem.

9.8 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle
2. The elements of Integration and Lebesgue Meassure
3. A course on integration- Nicolas Lerner
4. General theory of Integration- Dr. E.W. Hobson
5. General theory of Integration- P.Muldowney
6. General theory of functions and Integration- Angus E.Taylor.

9.9 ANSWERS TO CHECK YOUR PROGRESS

1. See lemma 9.1

2. See section 9.3

3. See section 9.4

4. See lemma 9.4

5. See section 9.4

UNIT-10 APPLICATIONS TO CALCULUS

STRUCTURE

10.0 OBJECTIVE

In this unit we will learn and understand about Integration by parts and related theorems, Mean value theorems, Bonnet's mean value theorem.

10.1 INTRODUCTION

Calculus, originally called infinitesimal calculus or "the calculus of infinitesimals", is the mathematical study of continuous change, in the same way that geometry is the study of shape and algebra is the study of generalizations of arithmetic operations.

It has two major branches, differential calculus and integral calculus; the former concerns instantaneous rates of change, and the slopes of curves, while integral calculus concerns accumulation of quantities, and areas under or between curves. These two branches are related to each other by the fundamental theorem of calculus, and they make use of the fundamental notions of convergence of infinite sequences and infinite series to a well-defined limit

Infinitesimal calculus was developed independently in the late 17th century by Isaac Newton and Gottfried Wilhelm Leibniz.Today, calculus has widespread uses in science, engineering, and economics.

In mathematics education, calculus denotes courses of elementary mathematical analysis, which are mainly devoted to the study of functions and limits. The word calculus (plural *calculi*) is a Latin word, meaning originally "small pebble" (this meaning is kept in medicine). Because such pebbles were used for calculation, the meaning of the word has evolved and today usually means a method of computation. It is therefore used for naming specific methods of calculation and related theories, such as propositional calculus, Ricci calculus, calculus of variations, lambda calculus, and process calculus.

Modern calculus was developed in 17th-century Europe by Isaac Newton and Gottfried Wilhelm Leibniz (independently of each other, first publishing around the same time) but elements of it appeared in ancient Greece, then in China and the Middle East, and still later again in medieval Europe and in India.

Ancient

He ancient period introduced some of the ideas that led to integral calculus, but does not seem to have developed these ideas in a rigorous and systematic way. Calculations of volume and area, one goal of integral calculus, can be found in the Egyptian Moscow papyrus (13th dynasty, c. 1820 BC); but the formulas are simple instructions, with no indication as to method, and some of them lack major components.

From the age of Greek mathematics, Eudoxus (c. 408–355 BC) used the method of exhaustion, which foreshadows the concept of the limit, to calculate areas and volumes, while Archimedes (c. 287–212 BC) developed this idea further, inventing heuristics which resemble the methods of integral calculus.

The method of exhaustion was later discovered independently in China by Liu Hui in the 3rd century AD in order to find the area of a circle. In the 5th century AD, Zu Gengzhi, son of Zu Chongzhi, established a method[] that would later be called Cavalieri's principle to find the volume of a sphere.

Medieval

In the Middle East, Hasan Ibn al-Haytham, Latinized as Alhazen (c. 965 – c. 1040 CE) derived a formula for the sum of fourth powers. He used the results to carry out what would now be called an integration of this function, where the formulae for the sums of integral squares and fourth powers allowed him to calculate the volume of a paraboloid

In the 14th century, Indian mathematicians gave a non-rigorous method, resembling differentiation, applicable to some trigonometric functions. Madhava of Sangamagrama and the Kerala School of Astronomy and Mathematics thereby stated components of calculus. A complete theory encompassing these components is now well known in the Western world as the *Taylor series* or *infinite series approximations*. However, they were not able to "combine many differing ideas under the two unifying themes of the derivative and the integral, show the connection between the two, and turn calculus into the great problem-solving tool we have today"

In Europe, the foundational work was a treatise written by Bonaventura Cavalieri, who argued that volumes and areas should be computed as the sums of the volumes and areas of infinitesimally thin cross-sections. The ideas were similar to Archimedes' in *The Method*, but this treatise is believed to have been lost in the 13th century, and was only rediscovered in the early 20th century, and so would have been unknown to Cavalieri. Cavalieri's work was not well respected since his methods could lead to erroneous results, and the infinitesimal quantities he introduced were disreputable at first.

The formal study of calculus brought together Cavalieri's infinitesimals with the calculus of finite differences developed in Europe at around the same time. Pierre de Fermat, claiming that he borrowed from Diophantus, introduced the concept of adequality, which represented equality up to an infinitesimal error term.The combination was achieved by John Wallis, Isaac Barrow, and James Gregory, the latter two proving the second fundamental theorem of calculus around 1670.

The product rule and chain rule,the notions of higher derivatives and Taylor series, and of analytic function[d]were used by Isaac Newton in an idiosyncratic notation which he applied to solve problems of mathematical physics. In his works, Newton rephrased his ideas to suit the mathematical idiom of the time, replacing calculations with infinitesimals by equivalent geometrical arguments which were considered beyond reproach. He used the methods of calculus to solve the problem of planetary motion, the shape of the surface of a rotating fluid, the oblateness of the earth, the motion of a weight sliding on a cycloid, and many other problems discussed in his *Principia Mathematica* (1687). In other work, he developed series expansions for functions, including fractional and irrational powers, and it was clear that he understood the principles of the Taylor series. He did not publish all these discoveries, and at this time infinitesimal methods were still considered disreputable.

These ideas were arranged into a true calculus of infinitesimals by Gottfried Wilhelm Leibniz, who was originally accused of plagiarism by Newton is now regarded as an independent inventor of and contributor to calculus. His contribution was to provide a clear set of rules for working with infinitesimal quantities, allowing the computation of second and higher derivatives, and providing the product rule and chain rule, in their differential and integral forms. Unlike Newton, Leibniz paid a lot of attention to the formalism, often spending days determining appropriate symbols for concepts.

Today, Leibniz and Newton are usually both given credit for independently inventing and developing calculus. Newton was the first to apply calculus to general physics and Leibniz developed much of the notation used in calculus today. The basic insights that both Newton and Leibniz provided were the laws of differentiation and integration, second and higher derivatives, and the notion of an approximating polynomial series. By Newton's time, the fundamental theorem of calculus was known.

When Newton and Leibniz first published their results, there was great controversy over which mathematician (and therefore which country) deserved credit. Newton derived his results first (later to be published in his *Method of Fluxions*), but Leibniz published his "Nova Methodus pro Maximis et Minimis" first. Newton claimed Leibniz stole ideas from his unpublished notes, which Newton had shared with a few members of the Royal Society. This controversy divided English-speaking mathematicians from continental European mathematicians for many years, to the detriment of English mathematics. A careful examination of the papers of Leibniz and Newton shows that they arrived at their results independently, with Leibniz starting first with integration and Newton with differentiation. It is Leibniz, however, who gave the new discipline its name. Newton called his calculus "the science of fluxions".

Since the time of Leibniz and Newton, many mathematicians have contributed to the continuing development of calculus. One of the first and most complete works on both infinitesimal and integral calculus was written in 1748 by Maria Gaetana Agnesi.

We will now apply the theory of the (generalized Riemann) integral that we have developed to obtain a number of results that are familiar from calculus, except that the hypotheses are much weaker than customary. This section is divided into four parts. In the first part, we will obtain very general versions of the Integration by Parts formula. We then apply these results to obtain various versions of the Mean Value Theorems. The third part is concerned with a theorem due to Hake that shows that the generalized Riemann integral does not admit (neither does it need) an extension analogous to the “improper integral” that is familiar from calculus. This result can also be viewed as providing a method for the evaluation of integrals. In the final part of this section new will consider integrands that depend on a parameter, and obtain some results that can be used in handling such integrals.

In this section new will limit our discussion to the case of a compact interval $I := [a, b]$. It will be seen later that most of these results can be extended to unbounded intervals.

Integration by parts

This familiar result is a consequence of the “product Rule” for differentiation. It will be convenient to use the notation

$$H\Big|_{\alpha}^{\beta} := H(\beta) - H(\alpha),$$

Where H is a function defined on an interval that contains the points α, β

We first consider the case where the functions $f, g \in R^{*}(I)$ have 0-primitives F,G, since this is the case most commonly encountered in elementary applications, and since the proof is very easy. We will show that the product FG is a c-primitive of the function $Fg + fG$ and that $F_g \in R^{*}(I)$ if and only if $f\,G \in R^{*}(I).$ In this case the familiar formula holds.

10.2 INTEGRATION BY PARTS

If $f, g \in R^{*}(I)$ have c-primitives F,G on an interval $I := [a,b], then\, Fg + fG$ has a c-primitive FG and therefore belongs to $R^{*}(I), and$ $(12.\alpha)$ $\int_a^b (Fg + fG) = FG\,|_a^b.$

Moreover, Fg belongs to $R^{*}(I)$ if and only if fG belongs to $R^{*}(I)$ in which case.

$(12.\beta)$ $\int_a^b Fg = FG\,|_a^b - \int_a^b fG.$

Proof. By hypothesis, $F\ and\ G$ are continuous on I and there exists countable sets $C_f\, and\, C_g\ of\ I$ such that

$F^{i}(x) = f(x)\ for\ x \in I - C_f\ and\ G^{'}(x) = g(x)\ for\ x \in I - C_{g.} Let\, C := C_f \cup C_{g,}$ so that C is a countable set. The Product Rule for differentiation implies that $(12.\gamma)$ $(FG)^{'}(x) = F(x)G^{'}(x) + F^{'}(x) = F(x)g(x) + f(x)G(x)$

For $x \in I - C.$ The Fundamental Theorem 4.7 implies that $Fg + fG = (FG)^{'}$ belongs to $R^{*}(I)$ and has integral $FG\,|_a^b, providing\,(12.\alpha).$

Theorem 3.1 implies that $Fg \in R^{*}(I)$ if and only if $fG \in R^{*}(I).$ Equation $(12.\beta)$ now follows from $(12.\alpha).$ Q.E.D.

We now present a theorem that gives a definitive form of the Integration by Parts formula in terms of indefinite integrals

$F(x) := \int_c^z f \, and \, G(x) := \int_c^x g \, of \; f, g \in R^*(I),$ rather than c-primitives of thee functions. This proof. Taken from [P-1;p.110] is considerabley more involved than that of 10.1

- 10.2 Integration by parts *. Let $f, g \in R^*(I)$ and let F,G be their respective indefinite integrals with base point $c \in I$.

(a) Then $Fg + fG$ belongs to $R^*(I)$ and has FG as indefinite integral with base point c. Therefore equation $(10.\alpha)$ holds.

(b) Moreover, Fg belongs to $R^*(I)$ if and only if fG belongs to $R^*(I)$ in which case equation $(10.\beta)$ holds.

Proof. We will treat the case where $c = \alpha$, leaving the general case as an exercise.

Theorem 4.11 implies that the indefinite integrals F,G are continuous and hence bounded on $I := [a, b]$ and we let $M \geq b - a > 0$ be such that $|F(x)| \leq M and |G(x)| \leq M \; for \, x \in I. Given \, \varepsilon > 0,$ we conclude from the continuity of F and G that there exists a gauge $\delta_{\in} on I$ such that if $x \in I, |x - t| \leq \delta_\varepsilon(t), then$

$$(12, \delta) | f(t)||.|G(t) - G(x)| \leq \varepsilon / 4M \; and \, | g(t)||.|F(t) - F(x)| \leq \varepsilon / 4M.$$

Since $f, g \in R^*(I),$ we may also assume that the gauge δ_ε is such that if $P := \{(|x_i - 1, x_i|t_i)\}_{i=1}^b \, is \, a \, \delta_\varepsilon -$ fine partition of $I,$ then

$$| S(f; P) - F(b)| \leq \varepsilon / 8M \text{ and } | S(g; P) - G(b)| \leq \varepsilon / 8M$$

It follows from Corollary 5.4 of the Saks-Henstock Lemma that

$$(12.\varepsilon_1) \quad \sum_{i=1}^n | f(t_i)(x_i - x_{i-1)} - [F(x_i) - F(x_i - 1)]| \leq \varepsilon / 4M,$$

$$(12.\varepsilon_2) \quad \sum_{i=1}^n | g(t_i)(x_i - x_{i-1)} - [G(x_i) - G(x_i - 1)]| \leq \varepsilon / 4M.$$

But, since we have

Notes

$$F(x_i)G(x_i) - F(x_{i-1})G(x_{i-1})$$

$$= F(x_i)G(x_i) - G(x_{i-1}) + G(x_{i-1})[F(x_i) - F(x_{i-1})]$$

And $= F(x_0) = 0 = G(x_0), we\, can \exp and\ F(b)G(b)$ in a telescoping sum:

$$F(b)G(b) = \sum_{i=1}^{n}[F(x_i)G(x_i) - F(x_{i-1})G(x_{i-1})]$$

$$= \sum_{i=1}^{n}\{F(x_i)[G(x_{i-1}) + G(x_{i-1})[F(x_i) - F(x_{i-1})]\}$$

Using the above expression, we obtain

$$\left|S(Fg + fG; \dot{P}) - F(b)G(b)\right|$$

$$= \sum_{i=1}^{n}\{F(x_i)[G(x_{i-1}) + G(x_{i-1})[F(x_i) - F(x_{i-1})]\}$$

$$(12.\varsigma_1) \quad \leq \sum_{i=1}^{n}\left|[F(t_i)g(t_i) + f(t_i)](x_i - x_{i-1}) - F(b)G(b)\right|$$

$$(12.\varsigma_2) \quad + \sum_{i=1}^{n}\left|[f(t_i)G(t_i)(x_i - x_{i-1}) - G(x_i - 1)[F(x_i) - F(x_{i-1})]\right|$$

But, since $F(t_i) = F(x_i) + [F(t_i) - F(x_i)], the\, term\, (12.\varsigma_i)$ is dominated by

$$(12.\eta_1) \quad \sum_{i=1}^{n} | F(x_i) | . | g(t_i)(x_i - x_{i-1}) - | G(x_i) - G(x_{i-1})] |$$

$$(12.\eta_2) \quad + \sum_{i=1}^{n} | g(t_i) | . | F(t_i) - F(x_i) | .(x_i - x_i - 1).$$

We now use the fact that $| F(x_i) | \leq M\ and (12.\varepsilon_2) in (12.\eta_1),$ and the second inequality in $(12.\delta)$ in $(12.n_2)$ to conclude that $(12.\varsigma_1)$ is dominated by

$$M.\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M}(b-a) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Using $G(t_i)=G(x_{i-1})+[G(t_i)-G(x_{i-1})]$, a similar argument shows that $(12.\varsigma_2)$ is also dominated by $\frac{1}{2}\varepsilon.$ Therefore, we inter that if $\dot{P}$ □ $\delta_{\varepsilon,}$ then

$$|S(F_g+fG;\dot{P})-F(b)G(b)|\leq\varepsilon.$$

But since $\varepsilon>0$ is arbitrary, we conclude that $Fg+fG$ belongs to $R^*(I)$ with integral $F(b)G(b).$

Since b can be replaced by an arbitrary point $x\in I,$ it follows that $Fg+fG\, has\, FG$ as indefinite integral with base point α .

The assertion in part (b) follows as before. Sometimes it is convenient to write formula $(12,\beta)$ in the "calculus form":

$$(12.\theta)\quad \int_a^b F(x)g(x)dx=FG\Big|_a^b-\int_a^b f(x)G(x)dx.$$

We recall that in calculus courses one often uses the notation

$$u(x):=F(x),\qquad d\upsilon(x):=g(x)dx-G^{'}(x)dx,$$

So that we have

$$d\upsilon(x)=F^{'}(x)dx=f(x)dx,\quad v(x)=G(x).$$

Hence equation $(12.\theta)$ takes the form

$$\int_a^b u(x)d\upsilon(x)=u(x)\upsilon(x)\Big|_a^b\int_a^b\upsilon(x)du(x),$$

Which is often abbreviated as

$$\int_a^b ud\upsilon=u\upsilon\Big|_a^b-\int_a^b\upsilon du.$$

The reader is certainly familiar with the technique of integrating by parts, so we will not given any routine examples here. Our first example shows that formula $(12.\beta)$ does not hold for the Lebesgue integral unless it is assumed that both $Fg\, and\, fG$ belong to $L(I).$ The second example shows that $(12.\beta)$ does not hold unless at least one (and hence both) of

$Fg\ and\ fG$ belongs to $R^*(I)$. The third example applies Theorem 10.2 to an interesting integral.

10.3 Examples. (a) it is possible that $Fg \in L(I)$. but $fG \notin L(I)$.

Let $F(x) := x\ and\ let\ G(x) : -x\cos(\pi / x^2)\ for\ x \in (0,1]\ and\ G(0) := 0.$ Then F is a primitive of $f(x) : -1\ and\ G$ is e-primitive on $I := [0,1]$ with exceptional set $\{0\}$ of the function.

$$g(x) := \cos(\pi / x^2) + 2\pi / x^2)\sin(\pi / x^2)\ for\ x \in (0,1]$$

And $g(0) := 0.$ Moreover, the product FG is a primitive of the function

$(FG)'(x) = 2x\cos(\pi / x^2) + (2\pi / x)\sin(\pi / x^2)$ for $x \in (0,1]$ And $(FG)'(0) = 0.$ In this case the product fG belongs to $L(I)$.

However, Fg has the form

$$(Fg)(x) = x\cos(\pi / x^2) + (2\pi / x)\sin(\pi / x^2) \quad \text{for} \quad x \in (0,1].$$

Now the first term belongs to $L(I)$, but it is seen i Exercise 10.C that the second term does not belong to $L(I)$. Therefore $Fg \notin L(I)$; however, both $Fg\ and\ fG\ belong\ to\ R^*(I)$.

(c) It is possible that neither of the functions $Fg\ and\ fG$ belongs to $R^*(I)$.

Let F, G be defined on $I := [0,1]\ by\ F(x) := 0 =: G(0)\ and$

$$F(x) := x^{1/2}\sin(\pi / x), \quad G(x) := x^{1/2}\cos(\pi / x) \quad \text{for} \quad x \in (0,1].$$

Then both F, G are continuous on $I := [0,1]$; moreover

$$F'(x) = \frac{1}{2}x^{-1/2}\sin(\pi / x) - \pi x^{-3/2}\cos(\pi / x)\ \text{for}\ x \in (0,1]$$

So that F is a c-primitive of $f := F'(where\ f(0) := 0)$ with exceptional set $\{0\}$. Therefore, if $x \in (0,1]$, then we have

$$f(x)G(x) = \frac{1}{2}\sin(\pi/x)\cos(\pi/x) - (\pi/x)\cos^2(\pi/x)\,,$$

Where we have used the identities

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta, \cos^2\theta = \frac{1}{2}(\cos 2\theta + 1).$$

Now the first term is bounded and measurable and hence is in $L(I)$, and the second term is seen to be in $R^*(I)$.The fact that $Fg \notin R^*(I)$ follows from this fact, or can be proved in the same way. (See Exercise 10.D.)

Thus the products $fG\ and\ Fg$ do not belong to $R^*(I)$ even though their sum does.

(c) If $f \in R^*([a,b])$, we will show that

$$(12,i)\quad \frac{1}{n}\int_a^b f(x)\cos n.xdx \to 0 \quad as\ n \to \infty,$$

and similarly if sin nx is replaced by sin nx (see Exercise 10.E). To prove the first assertion, we will make use of the Riemann-Lebesgue Lemma 9.17 that if $\varphi \in L([a,b])$, then

$$\int_a^b \varphi(x)\sin nx\,dx \to 0 \quad as\ \ n \to \infty,$$

And similarly if sin nx is replaced by cos nx.

To prove $(12,i)$,we let $F(x) := \int_\alpha^x f(x)dx \quad and\ G(x) := \int_\alpha^x \sin nx\,dx.$

Since F is continuous and $g(x) := G(x) = \sin nx, then\, Fg$ is continuous and belongs to $R^*([a,b])$. Theorem 10.2 then implies that $fG \in R^*([a,b])$ and that

$$\frac{1}{n}\int_a^b f(x)\cos nxdx = \frac{1}{n}F(b)\cos nb + \leftarrow \int_a^b F(x)\sin nx\,dx.$$

The Riemann-Lebeesgue Lemma applied to $\varphi = F\ now\, gives\,(12.i)$.

10.3 MEAN VALUE THEOREMS

We will now establish the important Mean Value Theorems in a high level of generality. In order to prove the Second Mean Value Theorem, we need to know that certain products of functions are integrable. In particular, we will use the Multiplier Theorem 10.10, which asserts that the product of a function in $R^*(I)$ and a function in BV(I) is integrable. For information concerning the history of these results, see Hobson [Hb-1;p.616ff].

- 10.4 First Mean Value Theorem. If f is continuous on $I := [a,b]$ *and if* $p \in R^*(I)$ does not change sign on I, then there exists $\xi \in I$ such that

(10.k) $\int_a^b fp = f(\xi)\int_a^b p.$

Proof. In fact, $p \in L(I)$ so that $fp \in L(I)$, *If* $p \geq 0,$ then $mp \leq fp \leq Mp,$ *where* $m := \inf(f(x) : x \in I \}$, and $M :-\sup\{ f(x) : x \in I \},$ so that

$$m\int_a^b p \leq \int_a^b fp \leq M\int_a^b p.$$

If $f_I\, p = 0$,he result is trivial; if not, it follows immediately from the Bolzano Intermediate Value Theorem. If $p \leq 0,$ the argument is similar.

Q.E.D

∘ 10.5 Second Mean Value Theorem. If $f \in R^*(I)$ *and* g is monotone on $I := [a,b],$ then there exists $\xi \in I$ such that

$$(12.\lambda) \quad \int_a^b fg = g(a)\int_a^\xi f + g(b)\int_\xi^b f.$$

Proof. It follows from the Multiplier Theorem 10.10 that $fg \in R^*(I),$ and from the Integration by Parts formula $(H.\gamma)$ for the Riemann-Stieltjes integral that

$$\int_a^b fg = \int_a^b g dF = gF\,|_a^b - \int_a^b f.$$

If we apply the Mean Value Theorem for the Riemann-Stieltjes integral (Theorem H.6), we conclude that there exists $\xi \in I$ such that the term on the right equals

$$gF\,|_a^b - F(\xi)g\,|_a^b = g(a)[F(\xi) - F(a)] + g(b)[F(b) - F(\xi)].$$
$$= g(a)\int_a^\xi f + g(b)\int_\xi^b f.$$

If we combine these expressions, we obtain $(12.\lambda)$. Q.E.D.

Proofs of special cases of this theorem not using the Riemann-Stieltjes integral are outlined in the exercises.

10.4 BONNET'S MEAN VALUE THEOREM

If $f \in R^*(I)$ *and* $g \geq 0$ is increasing on I, then there exists $\xi \in I$ such that $(12.\mu)$ $\int_a^b fg = g(b)\int_\xi^b f.$

Proof. Define $g_1 : I \to \square$ by $g_1(a) := 0$ *and* $g_1(x) := g(x)$ *for* $x \in (a,b]$.
Now apply the Second Mean Value Theorem 10.5 Q.E.D

There are analogous forms of Bonnet's Theorem for decreasing and for negative functions (see the exercises).

10.7 Examples. (a) If f is not continuous, then the First Mean Value Theorem 10.4 may fail.

Let

$f(x) := -1$ *for* $x \in| -1,0)$ *and* $f(x) := 1$ *for* $x \in [0,1]$ *and letp* $:= 1$ *on* $I := [-1,1]$. Then f is not continuous at 0, but $p > 0$ *and* $f\ p \in R^*(I)$. However, $f_1 fp = 0$ *and* $f_I p = 2,$ *so that* $(12.k)$ does not hold.

(b) If p changes sing, then the First Mean Value Theorem 10.4 may fail.

Indeed, let $f(x) := x =: p(x)$ *on* $I := [-1,1]$, so that f is continuous on I and $p, fp \in R^*(I)$. However,

$f_I fp = 2/3$ *and* $f_I p = 0,$ *so that* $(12.k)$ *fails.*

(c) If g is not monotone, the Second Mean Value Theorem 10.5 may fail. Let $f(x) := x^2 - 1 =: g(x)$ *onI* $:= [-1,1]$. Then $f \in R^*(I)$ and, although g is not monotone, $g \in BV(I)$. However, $f_I fg = f_I (x^2-1)^2 dx = 16/15$ while $g(-1) = g(1) = 0,$ *so that* $(12.\lambda)$ does not hold.

(d) If g is not increasing, then Bonnet's Theorem 10.6 may fail.

Let f and g be as in (c).

(e) If $0 > a < b,$ *then* $|\int_a^b x^{-1} \sin x dx| \leq 2(1/a + 1/b)$.

(f) If $g(x) := x^{-1}$ *on* $I := [a,b]$, *then* g is monotone on I, so the Second Mean Value Theorem 10.5 implies there exists ξ such that

$\int_a^b x^{-1} \sin x\, dx = (1/a)\left|\cos a - \cos\xi\right| + (1/b)[\cos\xi - \cos b]$. whence the inequality follows.

Notes

(g)The inequality in (e) (and the Cauchy Condition for the limit) establishes the existence of the important limit:

$$(12.v) \quad \lim_{T-\infty} \int_0^T \frac{\sin x}{x} dx = \int_0^\infty \frac{\sin x}{x} dx.$$

Check Your Progress

1. Prove: If $f, g \in R^*(I)$ have c-primitives F,G on an interval $I := [a,b], then\, Fg + fG$ has a c-primitive FG and therefore belongs to $R^*(I), and$ $(12.\alpha)$ $\int_a^b (Fg + fG) = FG|_a^b$. Moreover, Fg belongs to $R^*(I)$ if and only if fG belongs to $R^*(I)$ in which case.

$$(12.\beta) \quad \int_a^b Fg = FG|_a^b - \int_a^b fG.$$

2. Prove: Let $f, g \in R^*(I)$ and let F,G be their respective indefinite integrals with base point $c \in I$.

(a) Then $Fg + fG$ belongs to $R^*(I)$ and has FG as indefinite integral with base point c. Therefore equation $(10.\alpha)$ holds.

Moreover, Fg belongs to $R^*(I)$ if and only if fG belongs to $R^*(I)$ in which case equation $(10.\beta)$ holds.

3. Prove: If $f \in R^*(I)\, and\, g \ge 0$ is increasing on I, then there exists $\xi \in I$ such that $\int_a^b fg = g(b)\int_\xi^b f$.

10.5 LET US SUM UP

1. If $f, g \in R^*(I)$ have c-primitives F,G on an interval $I := [a,b], then\, Fg + fG$ has a c-primitive FG and therefore belongs to $R^*(I), and$

$$(12.\alpha) \quad \int_a^b (Fg + fG) = FG|_a^b.$$

Moreover, Fg belongs to $R^*(I)$ if and only if fG belongs to $R^*(I)$ in which case.

$$(12.\beta) \quad \int_a^b Fg = FG|_a^b - \int_a^b fG.$$

2. Let $f, g \in R^*(I)$ and let F,G be their respective indefinite integrals with base point $c \in I.$

(b) Then $Fg + fG$ belongs to $R^*(I)$ and has FG as indefinite integral with base point c. Therefore equation $(10.\alpha)$ holds.

Moreover, Fg belongs to $R^*(I)$ if and only if fG belongs to $R^*(I)$ in which case equation $(10.\beta)$ holds.

3. If f is continuous on $I := [a,b] and\, if\; p \in R^*(I)$ does not change sign on I, then there exists $\xi \in I$ such that

$$\int_a^b fp = f(\xi)\int_a^b p.$$

4. If $f \in R^*(I) and\; g \geq 0$ is increasing on I, then there exists $\xi \in I$ such that $\int_a^b fg = g(b)\int_\xi^b f.$

10.6 KEY WORDS

Integration by parts

Mean value theorem

Bonnet’s mean value theorem

10.7 QUESTIONS FOR REVIEW

1. Explain about Mean value theorem

2. Prove: Bornet’s Mean value theorem

10.8 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle

2. The elements of Integration and Lebesgue Meassure

3. A course on integration- Nicolas Lerner

4. General theory of Integration- Dr. E.W. Hobson

5. General theory of Integration- P.Muldowney

6. General theory of functions and Integration- Angus E.Taylor

10.9 ANSWERS CHECK YOUR PROGRESS

1. See section 10.3
2. See section 10.3
3. See section 10.4

UNIT-11 IMPROPER INTEGRALS

STRUCTURE

11.0 OBJECTIVE

In this unit we will learn and understand about Improper integrals, Hakes theorem, Integrals with parameter and theorems.

11.1 INTRODUCTION

In mathematical analysis, an improper integral is the limit of a definite integral as an endpoint of the interval(s) of integration approaches either a specified real number, $\infty, -\infty$ or in some instances as both endpoints approach limits. Such an integral is often written symbolically just like a standard definite integral, in some cases with *infinity* as a limit of integration.

By abuse of notation, improper integrals are often written symbolically just like standard definite integrals, perhaps with *infinity* among the limits of integration. When the definite integral exists (in the sense of either the Riemann integral or the more advanced Lebesgue integral), this ambiguity is resolved as both the proper and improper integral will coincide in value.

Often one is able to compute values for improper integrals, even when the function is not integrable in the conventional sense (as a Riemann integral, for instance) because of a singularity in the function or because one of the bounds of integration is infinite.

We will now prove a remarkable result that was established for the Perron integral by he in rich Hake in 1921. In effect it asserts that there is no such thing as an "improper integral" for the generalized Riemann integral. By this we mean that any function that has an "improper integral" is already integrable. However, the limiting procedure may be useful in evaluating the integral, as we will see.
We will state this result only for the case of the right endpoint. We leave it to the reader to formulate the statement from the left endpoint, or where the difficulty occurs at an interior point of the interval. In Section 16 we will establish Hake's Theorem for infinite intervals in R.

11.2 HAKE'S THEOREM

Let $I := [a,b] and\ f : I \to R.$ Then the function $f \in R^*(I)$ if and only if there exists $A \in R$ such that for every $c \in (a,b)$ the restriction of $f\ to [a,c]$ is integrable. Moreover, by Theorem 4.11 (see also Theorem 5.6),the indefinite integral of f with base point c is continuous at b, so that

$$\int_a^b f = \lim_{c \to b-} \int_a^c f.$$

Hence the statement follows with $A := f_I f$.

$(\Leftarrow)$ Suppose there exists $A \in R$ such that for every $c \in (a,b)$ the restriction of f belongs to $R^*[(a,c)] and (12, \xi)$ holds. Now let $(c_k)^{\infty}{}_{k=0}$ be a strictly increasing sequence with

$a = c_0 and\ b = \lim_k c_k . Given \in > 0, let\ r \in N$ be such that

$b - c_r \leq \varepsilon / (|\ f(b)| + 1)$ and such that if $t \in [c_r, b),$ then

$$\left| \int_a^t f - A \right| \leq \varepsilon.$$

If $k \in N, let \delta_k$ be a gauge on $I_k := [c_k - 1, c_k]$ such that if P_k is any δ_k fine partition of I_k then.

$$\left| S(f; P_k) - \int_{I_k} f \right| \le \varepsilon / 2^k.$$

Without loss of generality, we may assume that

(i) $\delta_i(c_0) \le \frac{1}{2}(c_1 - c_0),$ *and if* $k \ge 1,$ *that*

(ii) $\delta_{k+1}(c_k) \le \min\left\{ \delta_k(c_k), \frac{1}{2}(c_k - c_{k-1}), \frac{1}{2}(c_{k+1} - c_k) \right\}.$

(iii) $\delta_{k+1}(t) \le \min\left\{ \frac{1}{2}(t - c_{k-1}), \frac{1}{2}(c_k - t) \right\}$ *for* $t \in (c_{k-1,} c_k).$

We now define $\delta on\, I\, by$

$$\delta(t) := \begin{cases} \delta_k(t) & if\ t \in [c_{k-1,} c_k), \\ b - c_r & if\ t = b. \end{cases} k \in N,$$

Thus δ is a gauge on $I\ and\ we\ let\ \dot{P} := \left\{ (|[x_{i-1,} x_t], t_i) \right\}_{i=1}^{n}$ be a δ-fine partition of $I.$ Since b does not belong to any interval $I_k,$ the last subinterval $[x_{n-1,} b]\, in\, \dot{P}$ must have its tag $t_n = b.$ But since $\dot{P}$ □ $\delta,$ this implies that

$$Q_1 := \dot{P} \cap [c_0, c_1], ..., \ \dot{Q}_{v-1} := \dot{P} \cap [c_a - 2, c_{a-1}], \dot{Q}_a := \dot{P} \cap [c_{0-1,} x_{n-1}].$$

Since such

Now let $\delta \in N$ be the smallest positive integer such that $x_{n-1} \le c_0,$ so that $r \le s. If\ k = 1, ..., s-1,$ then condition (iii) implies that the point c_k must be a tag for any subinterval in $t \in T$ that contains c_k .Using the right-left procedure, we may assume that the points $c_0, ..., c_{0-1}$ are also points in $\dot{P}$. We let

$$Q_1 := \dot{P} \cap [c_0, c_1], ..., \ \dot{Q}_{v-1} := \dot{P} \cap [c_a - 2, c_{a-1}], \dot{Q}_a := \dot{P} \cap [c_{0-1,} x_{n-1}].$$

Since each $\dot{Q}_k (k = 1, ..., s-1] is\, a\, \delta_k$ -fine partition of $I_{k,}$ then

$$\left| S(f; \dot{Q}_s) \int_{c_{s-1}}^{x_{n-1}} f \right| \le \varepsilon / 2^{\delta}.$$

If $\dot{Q}_s$ *is a* δ_s-fine sub partition of $I_{s,}$ the Saks-Henstock Lemma 5.3 implies that

$$\left| S(f;\dot{Q}_s) \int_{c_{s-1}}^{\dot{x}_{n-1}} f \right| \leq \varepsilon / 2^{\delta}.$$

If $\dot{Q}_s := \{([x_{n-1}, b], b)\}, then\, S(f; Q^b) = f(\dot{b})(b - x_{n-1}),$ whence it follows that $|S(f;\dot{Q}^b)| \leq |f(b)|(b - x_{n-1}) \leq \varepsilon. Since\, \dot{P} = \dot{Q}_1 \cup ... \cup \dot{Q}_{s-1} \cup \dot{Q}_s \cup \dot{Q}^b,$ we have

$$|S(f;\dot{P}) - A| = |\sum_{i=1}^{s} S(f;\dot{Q}_1) + S(f;\dot{Q}^b) - A|$$

$$\leq |\sum_{i=1}^{s} S(f;\dot{Q}_1) + \int_a^{zn-1} f| + |S(f;\dot{Q}^b)|$$

$$+ |\int_a^{zn-1} f - A| \leq 3\varepsilon.$$

But since $\in > 0$ is arbitrary, then $f \in R^*(I)$ with integral A. Q.E.D

We now give some rather straightforward applications of Hake's Theorem, where the limit s taken at the left endpoint.

Examples. (a) Consider the integral $\int_0^1 x^r\, dx\ for\, r \in R.$

Let $g_r(x) := x^r\ for\ x \in (0,1]\, and\ g_r(0) := 0. If r + 1 > 0, then\, g_r$ has the function $x \mapsto x^{r+1}/(r+1)$ as a c-primitive with exceptional set either $\{0\}\, or\, \theta.$ Hence $g_r \in R^*[(0,1)]$ and

(12.0) $\int_0^1 x^r\, dx_r = x^{r+1}/(r+1)|_0^1 = 1/(r+1)\ for\ r > 1.$

(b) consider the integral $\int_0^1 x^r\, dx\ for\ r \geq 1.$

The function $x \mapsto x^{-1} has\, x \mapsto in\, x$ as a primitive on any interval [c,1] with $0 < c < 1.$ But since

$$\int_0^1 x^{-1}\, dx = In\, x|_c^1 = In\, c \to \infty$$

as $c \to 0+$, we conclude from Hake's Theorem that $x^{-1} \notin R^*([0,1]).$

Since $x^r < x\, when\, r > 1 and\ x \in (0,1), we\, have\, x^1 < x^{-r}$ so that $x^{-r} \notin R^*([0,1]),$ We give another proof of that fact here.

Indeed, $(1/x^2)\sin(\pi/x) = [(1/\pi)\cos(\pi/x)]' \; for \; x \in [c,1] with \, o < c < 1.$

thus we have

$$\int_c^1 (1/x^2)\sin(\pi/x)dx = (1/\pi)[\cos\pi \cos(\pi/c)].$$

Since $\cos(\pi/c)$ does not have a limit as $c \to 0+,$ it follows from Hake's Theorem that $(1/x^2)\sin(\pi/x) \notin R^*([0,1]).$

(d) Consider the integral $\int_c^1 (1/x^2)\sin(\pi/x)dx = (1/\pi)[\cos\pi - \cos(\pi/c)].$

Since $\cos(\pi/c)$ does not have a limit as $c \to 0+,$ it follows from Hake's Theorem that $(1/x^2)\sin(\pi/x) \notin R^*([0,1]).$

(c) Consider the integral $\int_0^1 x^r in\, x\, dx \; for\, r > -1.$

If s:=r+1>0, the integration by parts formula leads us to find that

$F(x) := s^{-1}[x^s In\, x - s^{-1}x^s] \;\; for \; x \in (0,1],$

And F(0):0 is a c-primitive with exceptional set {0}, where we have used L' Hospitals' Rule to show that F is continuous at x=0 when s>0.A calculation shows that if 0<c<1, then

$$\int_0^1 x^r in\, x\, dx = s^{-2}[c^s - 1] - s^{-1}c^s \, In\, c.$$

Another application of L 'Hospital's Rule shows that, if $r > -1,$ there is a limit as $c \to 0+, so\, x^r In\, x$ belongs to

$R^*([0,1]) \, and \; f_0^1 x^r In\, x\, dx = -(r+1)^{-2}.$

(e) Consider the integral $f_0^1 x^{-1} In\, x\, dx$

The function $G(x) = \frac{1}{2}(inx)^2 \; for\, x \in (0,1]$ has the property that

$G'(x) = x^{-1} In\, x \; for \; x \in (0,1].$ Hence ,if $0 < c < 1, then$

$$\int_c^1 x^1 in\, x\, dx = \frac{1}{2}(In\, c)^2 \to -\infty \; as\, c \to 0+.$$

We conclude from Hake's Theorem that the function $x^1 in\, x$ does notbelong to $R^*([0,1]).$

11.3 INTEGRANDS WITH A PARAMETER

We now consider integrals where the integrand depends on a parameter. For the sake of simplicity, we will treat the case where the domain of the

Notes

parameter is a bounded interval $T := [c, d]$, but many of our results can be extended to a consider ably more general parameter domain without difficulty.

The next several results make use of the following hypothesis:

Hypothesis (H).Let the function $f : I \times T \to \mathbf{R}$ be such that, for each $t \in T$, the function $x \mapsto f(x,t)$ is in R, in which case the function $F : I \to \mathbf{R}$ given by

(12.π) R

Is well defined. We want to show that various properties of $t \mapsto f(x,t)$ (e.g., limit, continuity, differentiability) carry over to similar properties of F.

The case where f (and $f_t = \partial f / \partial t$) are continuous on $I \times T$ is relatively familiar and will be outlined in the Exercises. However, it often happens that difficulties occur at the endpoints of the interval I. These difficulties are usually handled by assuming that the integral (12.π) converges uniformly with respect to $t \in T$. Arguments of this sort are to be found in many books dealing with this subject. In this section, we will treat the case where the hypotheses of joint continuity and uniform convergence are replaced by domination conditions.

- **Limit Theorem**. Let $f\ I \times T \to R$ satisfy Hypothesis (H), and suppose that:

(i) **There exists** $\tau \in T$ such that $f(x,\tau) = \lim i_{t-\tau} f(x,t)$ *for all* $x \in I$.

(j) There exist functions $\alpha, \omega R^*(I)$ such that $\alpha(x) \le f(x,t) \le w(x)$ for all $x \in I, t \in T$.

Then the function F in (12.π) exists on T and $F(\pi) = \lim_{g \to \tau} F(t)$: that is (12.$p$) $\int_a^b f(x,\tau) = \lim_{t \to \tau} \int_a^b f(x,t) dx.$

Proof. Hypothesis (H), condition (j), and the Integrability Theorem 9.1 imply that $x \mapsto f(x,t)$ belongs to $R^*(I)$ for each $t \in T$. Hence the function F given in (12.π) is defined on T. Now let (t_n) be any sequence in T coverging to τ .If we let $\int_n (x) := f(x, t_n)$ and

$\bar{f}(x,\tau)\, for\, x \in I$, it follows from (i),(f), and the Dominated Convergence Theorem 8.8 that

$$F(\tau) = \int_a^b \bar{f}(x)dx = \lim_{n\to\infty} D(t_n).$$

But since (t_n) is an arbitrary sequence in T converging to τ, we inter that $F(r) = \lim_{t\to\tau} F(t).$ Q.E.D.

Continuity Theorem. Let $f : I \times T \to R$ satisfy Hypothesis (H) and suppose that:

(i') The function $t \mapsto f(x,t)$ is continuous on T for each $x \in I := [a,b]$.

(j') There exists functions $\alpha, \omega \in R^*(I) such\, that$

$$\alpha(x) \le f(x,t) \le \omega(x) \text{ for all } x \in I, t \in T.$$

Then the function $F : T \to R$ given by $(12.\pi)$ is continuous on T.

Proof. We apply the Limit Theorem 11.11 to each point in T. Q.E.D.

We now obtain a result showing that the derivative of F can be found as an integral of the partial derivative $f_t := \delta f / \delta t.$ Thus, one can differentiate $F\, in (12.\pi)$ by 'differentiating under the integral sign", provided the partial derivative f_t is dominated by an integrable function.

Differentiation Theorem. Let $f : I \times T \to R$ satisfy Hypothesis (H) and suppose that:

(i") There exists $\tau \in T$ such that the function $x \mapsto f(x,\tau) i \sin R^*(I)$.

(j") the partial derivative $f_t exissts\, on\, I \times T$.

(K^*) There exist $\alpha, \omega \in R^*(I)$ such that

$$\alpha(x) \le f_t(x,t) \le w(x) \quad for\, all\, x \in I, t \in T.$$

Then we conclude that

(a) The function $x \leftrightarrow f(x,t) is\, in\, R^*(I)\, for\, each\, t \in T.$

(b) The function $x \leftrightarrow f_t(x,t) is\, in\, R^*(I)\, for\, each\, t \in T.$

(c) The function $F\, in\, (12.\pi) is\, defined\, end\, differentiable\, on\, T\, and$

$$(12.\pi)\quad F(t) = \int_a^b f_t(x,t)dx \quad for\, all\, t \in T.$$

Proof. Let τ he is hypothesis (I^*). If $x \in T, t \notin \tau,$ are fixed, then it follows from (f^*) and the Mean Value Theorem of calculus that there exists a point $x = s(x,t,\tau)\, between$ t an $d\tau$ such that

$$f(x,t) -- f(x,\tau) = (t-\tau) f_t(x,s).$$

Than, if $t \geq \tau, then (k^{''}) imples\, that$

$$f(x,\tau) + (t-\tau)\alpha(x) \leq f(x,t) \leq f(x,\tau) + (t-\tau)\omega(x).$$

While if $t \leq r, then (k^{''})$ imples that

$$f(x,\tau) + (t-\tau)\omega(x) \leq f(x,t) \leq f(x,\tau) + (t-\tau)\omega(x).$$

From Hypothesis (H), the above inequalities, and the Intergrability Theorem 0.1, we conclude that for each $t \in T,$ the function $x \mapsto f(x,t)$ belongs to $R^*(I).$ This is conclusion (s).

Now let $t \in T$ be fixed and let (t_n) be any sequence in T with $t_n \notin t, t_n \to t.$ It follows from $(f^{''})$ that

$$f_t(x,t) = \lim_{n\to\infty} \frac{f(x,t_n) - f(x,t)}{t_{n-t}} \qquad for\, x \in I.$$

Hypothesis (H) and the Measurable Limit Theorem 9.2 imply that the function $x \mapsto f_i(x,t)$ is measurable on I. From $(k^{''})$ and the Integrability Theorem 9.1, we conclude that $x \leftrightarrow f_i(x,t) is\, in\, R^*(I), Since\, t \in T$ is arbitrary, statement (b) follows.

If $t \in T$ is fixed and (t_n) is as before, assumption $(f^{''})$ and another application of the Mean Value Theorem imply that

$$\frac{f(x,t_n) - f(x,t)}{t_n - t} = f_t(x,s_n)$$

Where $s_n = s_n(x,t_n,t)$ lies between $t_n and\, t, so\, (k^{''})$ implies that

$$\alpha(x) \leq \frac{f(x,t_n) - f(x,t)}{t_n - t} \leq \omega(\tau)$$

But since we have

$$\frac{F(t_n)-F(t)}{t_n-t}=\int_a^b \frac{f(x,t_n)-f(x,t)}{t_n-t}dx.$$

It follows from the Dominated Convergence Theorem 8.8 that

$$\lim_{n\to\infty}\frac{F(t_n)-F(t)}{t_n-t}=\int_a^b f_t(x,t)dx.$$

Since (t_n) is an arbitrary sequence converging to $t \, with \, t_n \notin t$, we infer that $F^{''}(t)$ exists and is given by $(12.\sigma)$.

We now wish to establish a version of the familiar Leibniz formula concerting teh differentiation of an integral, were the limits as well as the integrand depend on the parameter. If we recall that an indefinite intergral is easily shown to be differentiable only at a point of continuity of the integrand, the hypotheses of the next theorem do not seem excessive. (While it is necessary that f be defined on $I^* \times T$ for a sufficiently large interval I^*, for the sake of simplicity we state this result for f defined on $R\times T$.)

Lelbniz' Formula. Let $f:R\times T\to R$ satisfy:

$(f^{''})$ the function $(x,t)\mapsto f(x,t)$ is continuous on $R\times T$.

$(f^{''})$ The partial derivative $(x,t)\mapsto f_i(x,t)$ exists and is continuous on $R\times T$

$(k^{''})$ The functions $u,\upsilon:T\to R$ are differentiable on T.

Then the function $G:T\to R$ defined by

$$(12.\tau) \quad G(t)=\int_{w(t)}^{(t)} f(x,t)dx.$$

Exists and a differentiable on T. Moreover, its derivative is given by

$$(12.\tau) \quad G(t)=\int_{w(t)}^{(t)} f(x,t)dx - f(u(t),t)u^{'}(t)+f(\upsilon(t),t)-x^{'}(t).$$

Proof. Let $I^{''}=[A,B] be\, an\, \mathrm{int}erval\, in\, R\, such\, that$

$$A+1\le u(t)x(t)\le B-1 \quad for \;\; t\in T.$$

It follows from $(i^{''''})$ that Hypotehesis (H) and condition 11.13 $(i^{''''})$ are satisfied. It follows from $(j^{''''})$ that 11.13 $(j^{''''})$ is satisfied and that f_t is bounded on $I^*\times T$, whence 11.13 $(k^{''''})$ is also satisfied.

For convenience, let $T^{''}:=T\times I^*\times I^*, and\; define \Gamma T^*$ by

$$\Gamma(t,u,v)=\int_u^v f(x,t)dx.$$

For u, v fixed, Theorem 11.13 implies that the partial derivative Γ_t exists on T^* and equals

$$\Gamma_t(t,u,v) = \int_u^v f_t(x,t)dx.$$

An extension of Theorem 11.12 to two parameters, the boundedness of the partial derivative $f_t on T^*$ and Theorem 4.11, imply that Γ_t is continuous in $(t,u,v) in T^*$.In addition, it follows from (i''') and Corollary 4.10 that the partial derivatives.

$$\Gamma_u(t,u,v) = -f(u,t) \quad and\, \Gamma_u(t,u,v) = f(\upsilon,t)$$

Are also continuous in $(t,u,v) on T^*$. Consequently, we may apply the Chain Rule (see (B-2;p.361]) to conclude that G is differentiable on T and that

$$G'(t) = T_t(t,u(t),\upsilon(t)).1 + \Gamma_u(t,u(t),v(t))u'(t)$$
$$+\Gamma_v(t,u(t),v(t)).v'(t).$$

Hence formula $(12.v)$ follows. Q.E.D

We conclude this discussion with a result concerning the interchange of the order of integration. We will be content with a result that includes the hypothesis that the integrand in $(12.\varphi)$ below has a primitive in

$$t \in T = (c,d[,\ for\, each\, x \in I.$$

- 11.15 Integration Theorem. Let $g, \gamma : I \times T \to R$ be such that γ satisfies Hypothesis (H), and suppose that:

(i) There exists $\tau \in T$ such that the function $x \mapsto \gamma(x,\tau) is\, in\, R^*(I)$.

(j) The partial derivate $\gamma t(x,t) = g(x,t)\, for\, all\, x \in I, t \in T$.

(k) There exists $\alpha, \omega \in R^*(I)$ such that

$\alpha(x) \le g(x,t) \le \omega(x)\ for\, all\, x \in I, t \in T$.

Then the function $x \mapsto g(x,t)$ belongs to $R^*(I)$ for each $t \in T$ so that

$(12.\varphi)\quad G(t) := \int_a^b g(x,t)dx$

Is defined on T. Moreover, $G \in R^*(I)\, and$

$(12.\chi)\quad \int_c^d G(t)dt = \int_a^b \left\{ \int_c^d g(x,t)dt \right\} dx.$

Proof. We will apply the Differentiation Theorem 11.13 with f replaced by γ.The hypotheses given above concerning γ correspond exactly to the hypotheses in 11.13 concerning f. We conclude from 11.13(a) that the function $x \mapsto \gamma(x,t)\, is\, in\, R^{*}(I)\, for\, each\, t \in T = [c,d]$ and we let

$$(12.\psi) \quad \int_c^d G(t)dt = T(d) - \Gamma(c).$$

On the other hand, since $g(x,t) = \gamma t(x,t),$ the Fundamental Theorem 4.5 implies that $t \mapsto g(x,t)$ belongs to $R^{*}(T)$ for each $x \in I$ and that

$$\int_c^d g(x,t)dt = \gamma(x,d) - \gamma(x,c).$$

Since $x \mapsto \gamma(x,t)$ is in $R^{*}(I)$ for each $t \in T,$ it follows from the preceding formula that $x \mapsto \int_c^d g(x,t)dt$ belongs to $R^{*}(I)$. Moreover.

$$\int_a^b \left\{ \int_c^d g(x,t)dt \right\} dx = \int_a^b \{\gamma(x,d) - \gamma(x,c)\} dx = \Gamma(d) - \Gamma(c).$$

If we combine the last equation with $(12.\psi)$ .we obtain $(12.\lambda)$ Q.E.D

Exercises

11.A Write out the details of the proof that $(12.\varsigma_2)$ is dominated by

$$\frac{1}{2} \in .$$

11.B Prove Theorem 11.2 fro an arbitrary base point $c \in [a,b].$

11.C Show that the function $k(x) := (2\pi / x)\sin(\pi / x^2)\, for\, x \in (0.1]$ and k(0):-0, arising in Example 11.3(a), does not belong to $L([0.1]).$

[Hint: consider $H(x) := x^2 \cos(\pi / x^2)\, for\, x \in (0.1]$

11.D Show directly that the product Fg in Example 11.3(b) does not belong to $R^{*}([0.1]).$

11.E Show directly that the function $Fg + fG$ in Example 11.3 (b) does not belong to $R^{*}([0.1]).$

11.F Let $F(x) := x^2 \sin(1/x^4) and G(x) := x^2 \cos(1/x^4)\, for\, x \in (0.1]$ and $F(0) := 0 =: G(0).$ Show that $F and\, G$ are differentiable at every point of [0,1] and that $FG^{'} + F^{'}G$ belongs to $R^{*}([0,1]), but\, that\, FG^{'} and\, F^{'}G$ do not belong to $R^{*}([0,1]).$

11.G (a) Show that $\int_a^b \varphi(x)\cos nx dx \to 0\, as\, n \to \infty when\, \varphi \in L([a,b]).$

Notes

(b) Use the result in (a) to show that $(1/n)\int_a^b f(x)\sin nx \to 0$ as $n\to\infty$ when $f\in R^*([a,b])$.

11.H. If f is continuous on $I:=[a,b] and\ p\in R^*(I)$ with $p(x)>0$ a.e. on I, show that there exists $\xi\in(a,b)$ such that $\int_a^b fp = f(\xi)\int_a^b p.$

11.I Let $f,p:[a,b]\to$ be such that $f^{'}$ exists on $[a,b]$ that $f^{'}p, p\in L([a,b])$ and that p>0a.e. Then there exists $\xi\in(a,b)$ such that $\int_a^b f^{'}p = f^{'}(\xi)\int_a^b p.$

11.J(a) If $f\in R^*[a,b], if\ g$ is increasing on [a,b] and if $A\le g(a), g(b)\le B,$ show that there exists $\xi\in[a,b]$ such that $\int_a^b fg = A\int_\alpha^\xi f + B\int_\xi^b f.$

(b) Formulate and prove an analogous statement when g is decreasing on [a,b].

11.K Let $f\in R^*([a,b]).$ Establish the following versions of Bonnet's Theorem.

(a) If $g\le 0$ is increasing on [a,b], then there exists $\xi\in[a,b]$ such that $\int_a^b fg = g(a)\int_a^\xi f.$

Check Your progress

1. Prove : Let $I:=[a,b] and\ f:I\to R.$ Then the function $f\in R^*(I)$ if and only if there exists $A\in R$ such that for every $c\in(a,b)$ the restriction of $f\ to[a,c]$ is integrable.

____

____

____

2. Prove: Let $f\ I\times T\to R$ satisfy Hypothesis (H), and suppose that: **There exists** $\tau\in T$ such that $f(x,\tau)=\lim i_{t-\tau} f(x,t)\ for\ all\ x\in I.$

There exist functions $\alpha,\omega R^*(I)$ such that $\alpha(x)\le f(x,t)\le w(x)$ for all $x\in I, t\in T.$ Then the function F in $(12.\pi)$ exists on T and $F(\pi)=\lim_{g\to\tau} F(t):$ that is $(12.p)$ $\int_a^b f(x,\tau) = \lim_{t\to\tau}\int_a^b f(x,t)dx.$

3. Prove: Let $f: I \times T \to R$ satisfy Hypothesis (H) and suppose that:

(i') The function $t \mapsto f(x,t)$ is continuous on T for each $x \in I := [a,b]$.

(j') There exists functions $\alpha, \omega \in R^*(I) such\, that$

$\alpha(x) \le f(x,t) \le \omega(x)$ for all $x \in I, t \in T$.

Then the function $F: T \to R$ given by $(12.\pi)$ is continuous on T.

Prove differentiation theorem.

11.4 LET US SUM UP

1. Let $I := [a,b] and\ f: I \to R.$ Then the function $f \in R^*(I)$ if and only if there exists $A \in R$ such that for every $c \in (a,b)$ the restriction of $f\ to [a,c]$ is integrable.

2. Let $f\ I \times T \to R$ satisfy Hypothesis (H), and suppose that:

There exists $\tau \in T$ such that $f(x,\tau) = \lim i_{t-\tau} f(x,t)\ for\, all\, x \in I.$

There exist functions $\alpha, \omega R^*(I)$ such that

$\alpha(x) \le f(x,t) \le w(x)$ for all $x \in I, t \in T.$ Then the function F in $(12.\pi)$ exists on T and $F(\pi) = \lim_{g \to \tau} F(t):$ that is $(12.p)$

$$\int_a^b f(x,\tau) = \lim_{t \to \tau} \int_a^b f(x,t)dx.$$

3. Let $f: I \times T \to R$ satisfy Hypothesis (H) and suppose that:

(i') The function $t \mapsto f(x,t)$ is continuous on T for each $x \in I := [a,b]$.

(j') There exists functions $\alpha, \omega \in R^*(I) such\, that$

$\alpha(x) \le f(x,t) \le \omega(x)$ for all $x \in I, t \in T$.

Then the function $F: T \to R$ given by $(12.\pi)$ is continuous on T.

11.5 KEY WORDS

Improper integral

Hakes theorem

Integrals with parameter

Differentiation theorem

11.6 QUESTIONS FOR REVIEW

1. Explain about improper integrals
2. Prove Hakes theorem.
3. Prove Differentiation theorem.

11.7 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle

2. The elements of Integration and Lebesgue Meassure

3. A course on integration- Nicolas Lerner

4. General theory of Integration- Dr. E.W. Hobson

5. General theory of Integration- P.Muldowney

6. General theory of functions and Integration- Angus E.Taylo

11.8 ANSWERS TO CHECK YOUR PROGRESS

1. See section 11.2
2. See section 11.2
3. See section 11.3
4. See section 11.3

UNIT-12 SUBSTITUTION THEOREMS

STRUCTURE

12.1 OBJECTIVE

In this unit we will learn and understand about First substitution theorems, Second substitution theorems and integral-gauges.

12.2 INTRODUCTION

The substitution (or change of variables) theorems, which are consequences of the Chain Rule of calculus, are often useful in converting one integral into another. For example, the integral

$$\int_0^1 \frac{du}{\sqrt{1-u^2}} = Arc\sin 1 = \frac{1}{2}\pi$$

Can be converted (by the substitutions $u = y^2 and\, u = \sqrt{x})$ into the integrals

$$\int_0^1 \frac{2ydy}{\sqrt{1-y^2}}\, and \quad \frac{1}{2}\int_0^1 \frac{dx}{\sqrt{x-x^2}}.$$

Both of these substitutions are instances of the First Substitution Theorem, which can be summarized by the formula

$$\int_{\phi(a)}^{\phi(b)} f(u)du = \int_a^b f\left(\phi(x)\right)\phi'(x)dx,$$

Which is valid under certain hypotheses that will be stated.

Notes

The transformation in the direction $u \to \phi(x)$ is usually straightforward. But every equation can be read from right to left, as well as from left to right. So the First Substitution Theorem is also useful when we see that an integral has the form on the right side of the above equation for appropriate functions $f \; and \; \phi$ .For instance, we note that the integral

$$\int_0^1 \frac{2xdx}{1+x^2}$$

has this form where $f(u) := (1+u)^{-1} \, and \, u = \phi(x) : -x^2$. so the value of this integral can be seen to equal in 17. Usually, however, the appropriate substitution is not as obvious as in this case.

The first Substitution Theorem

We will now establish the validity of the above formula under some conditions that are useful in practice. Frequently, the differentiable function ϕ is assumed to be a strictly montone mapping of the interval $\phi(I)$, but for many applications that hypothesis is not satisfied, and we need to permit ϕ to be many to one. Of course, it is necessary that f be defined on the interval $\phi(I)$, which contains – but need not equal – the interval with end points $\phi(a) \, and \phi(b)$.

We will first state a theorem in the case where $f : J \to R$ has a-c primitive $FonJ, and \, \varphi : I \to \square$ has a c-primitive $\phi \, on \, I$. We will also suppose that ϕ is a countable-to-one mapping of $f \operatorname{int} o \, J$ in the sense that $\phi^{-1}(\{p\})$ is a countable set in I for each $p \in J$.

It is stressed that for the equality of the two integrals, it is not necessary to know the c-primitive $F \, of \; f$ explicitly, but merely to know that F exists (as it does, for example, when f is regulated).

12.2 FIRST SUBSTITUTION THEOREM: I

I. Let $I := [a,b] and \; J := [c,d]$ and suppose that:

(i) $f : J \to R$ has a c-primitive $F \; on \, J$.

(ii) $\varphi : I \to R$ has a c-primitive ϕ on $I \; and \phi(I) \subseteq J$.

(iii) ϕ is a countable-to-one mapping of $I \operatorname{int} o \, J$.

Then $(f \circ \phi)\varphi \in R^*(I) \, and \; f \in R^*\left(\phi(I)\right) : moreover,$

$$(13.\alpha) \quad \int_a^b (f \circ \phi).\varphi = (F \circ \phi)\big|_a^b = \int_{\phi(\alpha)}^{\phi(b)} f.$$

Proof. By hypothesis (i), F is continuous on J and there exists a countable set $C_f \subset J$ such that $P^*(u) = f(u)$ *for all* $u \in J - C_{f.}$ By hypothesis (i). ϕ is continuous on I and there exists a countable set $C_\varphi \subset I$ such that $\phi^{'}(x) = \varphi(x)$ for all $x \in I - C_\varphi$. Therefore $F \circ \phi$ is continuous on I. Hypothesis (k) implies that $\phi^{-1}(C_f)$ is a countable set in $I, so C := C_\varphi \bigcup \phi^{-1}(C_f)$ is a countable set in I. The Chain Rule (see [B-S; p.162]) implies that

$$(F \circ \phi)^{'}(x) = F^{'}(\phi(x)).\phi^{'}(x) = (f \circ \phi)(x).\varphi(x)$$

For all $x \in I - C.$ Therefore $F \circ \phi$ is a c-primitive of $(f \circ \phi):\varphi,$ so that

$(f \circ \phi):\varphi$ belongs to $R^*(I)$ and

$$\int_a^b (f \circ \phi).\varphi = (F \circ \phi)\big|_a^b = F(\phi(b)) - F(\phi(a)).$$

It follows from (i) that $f \in R^*(J)$. But since $\phi(f)$ is a compact interval in J. Corollary 3.8 implies that f is integrable on $\phi(I)$ and also on the compact interval with endpoints $\phi(a), \phi(b)$.

If $\phi(a) \le \phi(b),$ then we apply the fundamental Theorem 4.7 to the interval $[\phi(b)\phi(a)]$ implies that

$$\int_{\phi(a)}^{\phi(b)} f = F\big|_{\phi(a)}^{\phi(b)} = F(\phi(b)) - F(\phi(a)).$$

Hence $(13.\alpha)$ bolds in either case. Q.E.D

Remarks. (a) If F is a primitive of f *on* J (so that $C_J = 0$), then hypothesis (k) is not needed in 12.1.

(b) As mentioned at the beginning of this section , we sometimes use formula $(13.\alpha)$ in the opposite direction. That is, to integrate $\int_\alpha^\beta \bar{f}(x)dx,$ we sometimes find a substitution $x = \Omega(v)$ that makes $(\bar{f} \circ \Omega)(v)$ a relatively simple function of $v,$ and such that $\omega(v) := \Omega^{'}(v)$ is also simple. In this case he formula $(13.\alpha)$ can be read as

$$(13.\alpha') \quad \int_{\alpha}^{\beta} \bar{f}(x)dx = \int_{a}^{b} (\bar{f} \circ \Omega).\omega,$$

Where $a \, and \, b$ are numbers such that $\alpha = \Omega(a) \, and \, \beta = \Omega(b).$ Of course, in using this approach we still have to verify that the hypotheses of Theorem 12.1 are satisfied.

Examples. (a) consider the integral $\int_{-1}^{-3} 2x\sqrt{1+x^2}dx.$

We note that if we put

$f(u) := \sqrt{u} \; for \, u \geq 0, and \; u = \phi(x) = 1 - x^2 then \, \phi'(x) = 2x \; for \; x \in |-1.3|.$

Thus the integrand has the form $(f \circ \phi)(x).\phi'(x),$ where f has the primitive $F(u) := u^{3/2} and \, C_J = \theta.$ Since $\phi(-1) = 2 \, and \, \phi(3) = 10.$ Theorem 12.1 implies that

$$\int_{-1}^{3} 2x\sqrt{1+x^2}dx = \int_{2}^{10} \sqrt{u}du = \frac{2}{3}u^{2/2} \Big|_{u=2}^{u=10}$$

$$= \frac{2}{3}(10\sqrt{10} - 2\sqrt{2}).$$

Note that $\phi([-1,3]) = [0,10] \neq [\phi(-1), \phi(3)]$ and that ϕ is not one-one on $[-1,3];$ however, it is at most two-to-one on this interval.

The integrand becomes unbounded as $x \to 0+;$ hence there is some doubt about the existence of the integral. But, since the function is (measurable and) dominated in absolute value by $1/\sqrt{x} \in L([0,0]),$ the integral certainly exists.

We let $u = \phi(x) = \sqrt{x} \; on [0,9]$ so that ϕ is an f-primitive of $\psi(x) = 1/(2\sqrt{x}).$ If we put $f(u) := \cos u,$ which has a primitive $F(u) = \sin u,$ then the integrand has the form

$$f(\phi(x)).\varphi(x) = \frac{\cos\sqrt{x}}{2\sqrt{x}} \quad for \; x \in (0.9].$$

Since $C_j = \theta,$ condition (k) is satisfied. Theorem 12.1 now yields

$$\int_{0}^{9} \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int_{0}^{3} \cos u \, du = 2\sin u \, |_{0}^{3} = 2\sin 3.$$

(b') We consider the same integral as in (b).

This time we notice that since the integrand involves $\sqrt{x},$ it would be simplified if we used the substitution $x = \Omega(v) := v^2$, *for if* $v \geq 0$ then

$\sqrt{x} = \sqrt{x^2} = v$ and so the integrand.

$$\bar{f}(x) = \frac{\cos\sqrt{x}}{\sqrt{x}} \quad becomes \quad (\bar{f} \circ \Omega)(\upsilon) = \frac{\cos\upsilon}{\upsilon}$$

At least for $\upsilon > 0.$ Moreover $\omega(\upsilon) = \Omega'(v) = 2\upsilon \, so \, that$

$$(\bar{f} \circ \Omega)(\upsilon).\omega(\upsilon) = \frac{\cos\upsilon}{\upsilon} 2\upsilon = 2\cos\upsilon.$$

We also note that Ω is a one-one mapping of (0.3] onto [0.9]. so we are led once more to the integral $\int_0^3 2 \cos \upsilon \, dx$

Now since $\bar{f} \in R^*([0.9])$ is continuous on (0.9], it follows from Corollary 4.9 that it has an (unknown) f-primitive on (0.9]. Clearly Ω is a primitive of ω on (0.3] and Ω is one-one. Thus the hypotheses of 12.1 are satisfied.

(c) Consider the integral $\int_0^1 x(1-x^2)^{-1} dx.$

The integrand is unbounded as $x \to 1-,$ so we will first consider the integral over $I_\delta := (0.b], with \, b \in (0.1).$ If we let $u = \phi(x) := 1 - x^2,$ then ϕ is a primitive of $\varphi(x) := -2x \, on \, I_{b.}$ Further, ϕ is a strictly decreasing map of I_b onto $J_b := [1-b^2, 1]$ Also, the function

$f(u) := (2u)^{-1} \, for \, u \in J_b$ has a primitive $F(u) := \frac{1}{2} in \, u \, on \, J_b.Hence$

$$F(u) := \frac{1}{2} in \, u \, on \, J_b.Hence$$

$$\int_a^b \frac{xdx}{1-x^2} = -\int_0^b \frac{-2x}{2(1-x^2)} = -F \circ \phi \,|_0^b = -\frac{1}{2} 1n(1-b^2).$$

Now that the limit of $-\frac{1}{2}1n(1-b^2)$ does not exist in $\square \; as \, b \to 1-.$

Therefore, by Hake's theorem 12.8, the Hake's Theorem 12.8, the integral $\int_0^1 x(1-x^2)^{-1} \; dx$ does not exist.

(d) Consider the integral $\int_0^1 \sqrt{1-x^2 dx.}$

We will make use of $(13.\alpha')$. Our knowledge of the trigonometric functions suggests that we use the substitution

$$x = \Omega(v) := \sin v. \, If \;\bar{f}(x) = \sqrt{1-x^2}, then$$

$$(\bar{f} \circ \Omega)(v) = \sqrt{1-\sin^2 v} = \sqrt{\cos^2 v} = |\cos v|.$$

Since $\bar{f}$ is continuous on [0,1], it has a primitive on this interval. We note that Ω maps the interval $(0\frac{1}{2}\pi]$ in a one-one fashion onto (0.1] and is the primitive of $\omega(v) = \cos v \geq 0 \, on \, (0\frac{1}{2}\pi]$ Therefore equation $(13.\alpha')$ gives

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{1}{2}\pi} \cos v\,dv = \int_0^{\frac{1}{2}\pi} \cos^2 \upsilon\, d\upsilon$$

$$= \frac{1}{2}[\sin \upsilon \cos \upsilon + \upsilon]\Big|_0^{\frac{1}{2}\pi} = \frac{1}{2}[\sin(\frac{1}{2}\pi)\cos(\frac{1}{2}\pi) + (\frac{1}{2}\pi] = \frac{1}{2}\pi.$$

The Second Substitution Theorem

The way that substitutions are often approached in calculus courses is to observe that setting $u = \phi(x)$ will make it possible to write part of the integrand in the form $f \circ \phi$ and then hope that we can come up with the factor $\varphi = \phi'$ somewhere. If we can 'build up' the factor φ by adjusting the remaining part of the integrand by constants, then we can use Theorem 12.1 otherwise, we usually abandon this substitution and try another one.

However, there is another theorem that enables us to convert at integral involving $f \circ \phi$ into an integral involving f and the derivative of function $\varphi := \phi^{-1}$ that is inverse to ϕ. Since we require ϕ to have an inverse function, we now assume that ϕ is one –one.

First we recall that if $\phi : I \to \mathbb{R}$ is continuous on $I = [a,b]$ and has a nonzero detivative on [a,b], then by the Darboux Intermediate Value Theorem (see [B.S;p.174]) applied to compact subintervals of (a,b), we infer that the derivative $\varphi = \phi'$ has constant sign on (a,b). Hence, by the Mean Value Theorem and the continuity of ϕ, the function ϕ is strictly monotone on [a,b] and we therefore have either

$$\psi(u)=\phi^{'}(u)=\frac{1}{\phi^{'}(\psi(b))}=\frac{1}{\varphi(\psi(u))}$$

If $\psi=\psi^{'}$ does not exist at one or both of the endpoints $\phi(a),\phi(b)$, we define it to equal 0 there.

We now state a version of Second Substitution Theorem. Although it is not the most general theorem possible, it applies in very many circumstances. We have taken pains to allow the possibility that $\varphi=\phi^{'}$ vanishes (or does not exist) at the end points a,b, since this case often arises in applications.

$\circ$ 12.3 Second Substitution Theorem, I. Let $I:=[a,b] andJ:=[c,d]$ and suppose that :

$(f^{'})\, f:J\to R.$

$(f^{'})\phi:I\to R$ is continuous, $\phi(I)\subseteq J \; and \; \phi^{'}$ exists and is $\neq 0$ on (a,b).

$(k^{'})\, f\circ\phi$ is integrable on $I \; and \; f.\psi$ is integrable on $\phi(I), when \psi$ is the function inverse to $\phi \, and \, \psi=\Psi^{''} on\, \phi\big((a.b)\big).$ Moreover

$$(13.\beta)\quad \int_a^b f\circ\phi = W\,|_a^b = \int_{\phi(a)}^{\phi(b)} f.\psi$$

Remarks. (a) A 0-primitive W exists if f is regulated (and so, if it is either continuous or moonstone) on J

(b) Sometimes it is not easy to find a c-primitive W of $f\circ\phi,$ but one can find a 0-ptimitive of $f\varphi$.

(c) When these c-primitives are not known explicitly, one still has the equality of the integrals

Proof of 12.3. We note that hyperthesis $(f^{'})$ implies that there exists a continuous strictly monotone function ψ inverse to ϕ and that $\psi and \; that \, \psi(u)=\phi^{'}(u)$ exists for all $u\in\phi((a,b))\subset J.$

It W is & 0-primitive of $f\circ\phi$ on $I,$ there exists a countable set $C_1\subset I$ such that $W^{'}(x)=f(\phi(x)), for\; x\in I-C_{j.}$ WE let $C_2=\phi(C_1)\bigcup(\phi(a),\phi(b))$ so that C_2 is a countable set in the compact interval $\phi(I).$ By the Chain Rule, if $u\in\phi(I)-C_2, then\, u\notin\{\phi(a),\phi(b)\} and\, \Psi(u)\notin C_{1,}$ so that

$$(W\circ\phi)^{'}(u)=W^{'}(\psi^{'}(u)).\psi^{'}(u)=f(\phi\circ\psi(u)).\psi(u).\psi(u).$$

Thus $W \circ \psi$ is a c-primitive of $f.\psi \; on\, \phi(I), so\, that \; f.\psi \in R^*(\phi(I))$.

Moreover,

$$\int_a^b f \circ \phi = W\,|_a^b = (W \circ \psi)\,|_{\phi(a)}^{\phi(b)} = \int_{\phi(a)}^{\phi(b)} f.\psi.$$

Therefore equation $(13.\beta)$ is established. Q.E.D 12.4 Examples.

Consider the integral $\int_0^2 (1+\sqrt{x})^{-1} dx.$

WE use the Second Substitution theorem 12.3 with $f(u) := (1+u)^{-1}$ and $u = \phi(x) = \sqrt{x}.$ Here $\phi^{'}(x) = 1/(2\sqrt{x}) > 0 \; for\, x \in (0.2]$ and it is clear that $\psi(u) = u^2 \; so\, that\, \psi(u) = 2u.$ Note that f is continuous (and also monotone) on $\phi(0.2]) = (0, \sqrt{2}],$ so we have

$$\int_0^2 \frac{dx}{1+\sqrt{x}} = \int_0^2 (f \circ \phi)(x) dx$$

$$= \int_0^{\sqrt{2}} f(u) - \psi(u) du = \int_0^{\sqrt{2}} \frac{1}{1+u}.2udu.$$

If we let $u = (1+u) - 1$ on the right, we readily show that the value of this integral equals $2[\sqrt{2} - In(1+\sqrt{2})]$.

[Of course, in calcubas we were taught to set $u = \sqrt{x}$ so that $x = x^2$ and hence $dx = 2udu,$ whence

$$\frac{dx}{1+\sqrt{x}} = \frac{2udu}{1+u}$$

Thus the somewhat mysterious juggling with differentials gives the correct result even though it gives no indication that the point $x = 0$ is a difficult point for the function ϕ.]

(b) Consider the integral $\int_0^1 (1-\sqrt{x})^{-1} dx.$

The integrand is unbounded as $x \to 1-,$ so we will consider the integral over [0,b] with $b \in (0.1).$ As in (a), the substitution $u = \phi(x) := \sqrt{x}$ gives

$$\int_0^b \frac{dx}{1-\sqrt{x}} = \int_0^{\sqrt{b}} \frac{2udu}{1-u} = 2[u + In(1-u)]\,|_0^{\sqrt{b}}$$

$$= 2[\sqrt{b} + In(1-\sqrt{b})].$$

But this last expression approaches $+\infty as\, b \to 1-$, so Hake's Theorem 12.8 implies that the integral $f_0^1 (1-\sqrt{x})^{-1} dx$ does not exist.

An Extension of Theorem 12.1

We now give a version of the First Substitution Theorem 12.1 that does not assume the existence of the c-primitive of the function f. Instead, it will be assumed that the substitution function $\phi : I \to \mathbb{R}$ is continuous, strictly monotone, and differentiable except possibly on a countable set $R \subset I$.

12.3 FIRST SUBSTITUTION THEOREM II.

Let $I := [a,b] and J := [c,d]$ and let $f : J \to R. Let\, \phi : I \to R.$ be a continuous strictly monotone function with $\phi(I) \subseteq J$ and suppose that there exists a countable set $R \subset I$ such that $\phi(x) := \phi^{'}(x)$ exists for all $x \in I - C$ and let $\varphi(x) := 0 \; for\; x \in C.$

- (a) Then f belongs to $R^{*}(\phi(I))$ if and only if $(f \circ \phi).\varphi$ belongs to $R^{*}(I).$

In either case , we have

$$(13.\gamma) \quad \int_{\phi(a)}^{\phi(b)} f = \int_a^b (f \circ \phi).\varphi.$$

Remark. Formula $(13.\gamma)$ will be proved when ϕ is a strictly increasing function, but it also remains true when ϕ is strictly decreasing, as the reader may show. However, in that case, if $I := [a,b].$ then $\phi(I) = [\phi(b), \phi(a)],$ so the integral on the left side is from a larger to a smaller value. Bearing in mind that the derivative $\varphi = \phi^{'}\, is \leq 0\, on\, I.$ we can write both the increasing and the decreasing case in the form

$$(13.\gamma') \quad \int_{\phi(I)} f = \int_I (f \circ \phi).|\varphi|,$$

Which is consistent with the situation in higher dimensions.

The proof of Theorem 12.1 was based on the Chain Rule and was quite straightforward. The proof of Theorem 12.5 is more involved because we need to take a careful look at the Riemann sums of two functions on different intervals. This theorem was given by McLeod [McL;pp.64-64]

Notes

without a detailed proof, and proved by marie Bielawski by carefully adjusting the gauges for the Riemann sums approximating the two integrals in $(13.\gamma)$. We will modify her argument by the use of "interval-gauges", which provides an alternative approach to the generalized Riemann integral and so has some independent interest.

12.4 INTERVAL-GAUGES

In Exercises1.T we defined an interval-gauge on $I := [a,b]$ to be a mapping $t \mapsto \Delta(t)$ of points $t \in I$ into bounded closed intervals $\Delta(t) - [a,(t),b(t)]$ such that $t \in (a(t0 < b(t)) \; for\, all\, t \in I.$ We say that an interval-gauge $\Delta on\, I$ is symmetric if t is the midpoint of $\Delta(t)$ for all $t \in I.$ It is clear that if δ is a (point) gauge on $I,$ then we can define a symmetric interval-gauge

$$t \mapsto \Delta(t) := [t - \delta(t), t + \delta(t)]$$

Corresponding to δ.Conversely, if Δ is an interval-gauge on $I,$ then we can define a (point) gauge $\delta_\Delta on\, I$ by

$$(13.\delta) \quad \delta_\Delta(t) := \min\{t - a(t), b(t) - t\} \quad for\, t \in I.$$

As in Exercise 1.U, if Δ is an interval-gauge on $I,$ we say that a tagged partition $\dot{P} := \{(I_{i,}t_i)\}_{i-1}^{n}\, of\ I\, is\, \Delta\ fine\, if\ I_i \subseteq \Delta(t_i)\, for\, all\, i = 1,...,n.$If Δ is an interval-gauge and if we define δ_Δ as in $(13.\delta)$, then any partition $\dot{P}\, of\ I$ that is δ_Δ-fine is certainly also $\Delta -\ fine.$ Thus, by cousin's Theorem 1.4, for every interval-gauge Δ there exist tagged partitions that are $\Delta -\ fine.$

We say that a function $g : I \to R$ is integrable on I to a number $D \in \square$ if for every $\in > 0$ there exists an interval-guage $\Delta_\in$ on I such that if $\dot{P}$ is any $\Delta_\in$-fine partition of I, then $(S(g;\dot{P}) - D| \le \in$.

From the preceding remarks, we see that a function g is integrable in this sense if and only if it is integrable in the sense of Definition 1.7, and the values of the integrals are equal.

The reason why interval-gauges are useful is that they map nicely with respect to strictly monotone functions. In more detail. Let $\phi : I \to R$ be a continuous strictly increasing function. We extend ϕ to all of R by defining

$$\phi(x) := \begin{cases} \phi(a) + t - a & for\, t < a, \\ \phi(b) + t - b & for\, t > b. \end{cases}$$

This extended function ϕ is continuous and strictly increasing on R there-fore, it is an order-preserving map that sends compact intervals to compact intervals, and sends the endpoints (and the interior points) of such intervals to the endpoints (and the interior points) of the images of these intervals.

Now, if Δ is an interval-gauge on I, then we define Δ on $\phi(I)$ by

$$\Delta(\phi(t)) := \phi(\Delta(t)) \quad for \;\; t \in I.$$

The properties of ϕ mentioned above show that Δ is an interval-gauge on $\phi(I)$.. Moreover, if $s = \phi(t)$ and $\Delta(t) = [a(t), b(t))].$ then

$$\Delta(s) = [\phi(I_i), \phi(b(t))].$$

Similarly, if $\dot{p} := \{(I_i, t_i)\}_{i=1}^{n}$ is a tagged partition of I ,then we define

$$\dot{p} := \{(\phi(I_i)), \phi(b(t))\}_{i=1}^{n}.$$

The properties of ϕ imply that $\dot{P}$ is a tagged partition of $\phi(I)$.

Moreover , it is clear that if $\dot{P}$ is Δ fine, then $\dot{P}$ is Δ-fine.

On the other land, if $\psi : R \to R$ is the function inverse to ϕ, then ψ is also continuous and strictly increasing. Thus it maps an interval-gauge Γ on $I = \psi(\phi(I))$ given by

$$\Gamma(t) := \psi(I^{'}(\phi(t)) \quad for\, t \in I.$$

Similarly, if $\dot{Q} := \{(\psi(J_k), \psi(s_k))\}_{k=1}^{m}$ is a tagged partition of $\phi(I),$ then

$$Q := \{(\psi(J_k), \psi(s_k))\}_{k=1}^{m}$$

Is a tagged partition of I and if $\dot{Q}$ is a $\Gamma - fine$ partition of $\phi(I),$ then we see that $\dot{Q}$ is a $\Gamma - fine$ partition of I Further, if $\dot{Q}$ is given and $\dot{P} := \dot{Q},$ then $\dot{Q} = \dot{P}.$

Remark. An observant reader (with good eyesight) will have noticed that we are using acute accents to indicate the transformed interval-gauges and partitions from I into $\phi(I),$ and grave accents to indicate the transformed interval-gauges and partitions from $\phi(I)\, to\, I.$

The above discussion was for a strictly increasing function ϕ If ϕ is continuous and strictly decreasing, the we extend ϕ to all of R by defining is to have slop-1 outside of R and obtain a continuous and strictly decreasing function on R. This extended function is an order-reversing map of R Consequently, the endpoints of the intervals need to be reversed, but the preceding considerations are readily modified.

Proof of Theorem 12.5

We now show that the stated properties of ϕ guarantee that a suitably fine partition of I gives rise to Riemann sums of f and $(f \circ \phi).\varphi$ that are nearly equal. We will treate only the increasing case. The reader will note that the proof of this lemma is similar to that of the Fundamental Theorem 4.7.

12.6 Lemma. Let $f : J \to R$ and $\phi, \varphi : 1 \to R$ be as in Theorem 12.5. Given $\in > 0,$ there exists an interval-gauge Δ_ϕ on I such that if $\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ is any $\Delta_\phi - fine$ partition of I and if $\dot{P} := \{(\phi(I_i), \phi(t_i))\}_{i=1}^{n}$ then

(13. ∈) $$\left|S\left(f:\dot{P}\right)-S\left(f0\phi\right).\varphi;\dot{P})\right|\leq\varepsilon\left(b-a+1\right)$$

Proof. Let $C=\{c_k\}$ be an enumeration of the set where the derivative $\phi^{'}=\varphi$ does not exist and let $\in>0$ be given If $t\in I-C,$ then ϕ is differentiable at t, so we may apply the Straddle Lemma 44 to find a compact interval $\Delta\phi(t)$ with midpoint t such that if $u,\upsilon\in\Delta_{\phi}(t)\, and\, u\leq t\leq\upsilon,$ then

$$\left|\phi(\upsilon)-\phi(u)-\varphi\left(t\right)(\upsilon-u)\right|\leq\frac{\in(\upsilon-u)}{|f(\phi(t))|+1}$$

Whence it follows that

(13.ς) $$\left|f(\phi\left(t\right)|\phi(\upsilon)-\phi(u))]-f(\phi(t)).\varphi(t)(\upsilon-u)\right|\leq\in(\upsilon-u).$$

Since ϕ is continuous at $c_k\in I,$ there exists a compact interval $\Delta_{\phi}(c_k)$ with midpoint c_k such that if $u,\upsilon\in\Delta_{\phi}(c_k)\, and\, u\leq c_k\leq\upsilon,$ then

$$|\phi(\upsilon)-\phi(u)|\leq\frac{\in}{2^k(|f(\phi(c_k))|+1).}$$

and since $\varphi(c_k):=0,$ we have

(13.ς) $$\left|f\phi\left(c_k\right))|\phi(r)-\phi(u)]-f(\phi(c_k)).\varphi(c_k)(\upsilon-u)\right|\leq\frac{\in}{2^k}$$

.

Now let $I_i=|x_{i-1,}x_t],$ so that $\phi(I_i)=[\phi(x_{i-1}),\phi(x_i)].$ It follows that

$$S\left(f;\dot{P}\right)=\sum_{i-1}^{n}f(\phi(t_i))[\phi(x_i)-\phi(x_{i-1})].\text{ and}$$

$$S\left(f\circ\phi\right).\varphi;\dot{P})=\sum_{i=1}^{n}f(\phi(t_i)).\varphi(t_i)(x_i-x_{i-1}).$$

Therefore, if we use (13.ξ) and (13.π) ,then we conclude that

$$\left|S(f;\dot{P})-S\left((f\circ\phi)\right).\varphi;\dot{P})\right|<\in\sum_{i=1}^{n}(x_i-x_{i-1})+\sum_{k=1}^{\infty}\in/2^k$$

$$=\in(b-a)+\in,$$

Which is the stated inequality. Q.E.D

We are now prepared to give the proof of Theorem 12.5. We will consider only the case of increasing ϕ.

Proof of Theorem 12.5. (a) Suppose that $(f\circ\phi).\varphi$ belongs to $R^*(I)$ with integral A. Then, given $\in>0$ there exists an interval-gauge $\Delta_\in$ on I such that if $\dot{P}$ is a $\Delta_\in$-fine partition o f I, then $|S((f\circ\phi).\varphi;\dot{P})-A|\le\in$. If Δ_ϕ is the interval-gauge in Lemma 12.6, we let $\Delta_\in\bigcap\Delta_{\phi,}$ so that if $\dot{P}$ is a $\Delta_\in$-fine partition of I ,then it is both $\Delta_\in$-fine and Δ_ϕ-fine. Now let $\Delta_\in$ be the interval-gauge on $\phi(I)$ that corresponds to $\Delta_\in$.If $\dot{Q}$ is a $\Delta_\in$-fine partition of $\phi(I)$,then $\dot{P}$,so that $S(f;\dot{Q})=S(f;\dot{P})$. Therefore we have

$$\left|S(f;\dot{Q})-A\right|\le\left|S(f;\dot{P})-S(f\circ\phi).\varphi;\dot{P})\right|+|S((f\circ\phi).\varphi;\dot{P})-A|$$

$$\le\in(b-a+2).$$

Since $\in>0$ in arbitrary, we deduce that $f\ is\ in\ R^*(\phi(I))$ with integral A.

Now suppose that f belongs to $R^*(\phi(I))$ with intergral B. Then, given $\in>0$, there exists an interval-gauge $T_\in$ on $\phi(I)$ such that if $\dot{Q}$ is any $T_\in$-fine partition of $\phi(I)$, then $|S(f;\dot{Q})-B|\le\in$. Now let $T_\in$ be the interval gauge is an interval-gauge on I .Now, if $\dot{P}$ is an $\Omega_\in$-fine partition of I ,then $\dot{P}$ is $\Delta_\in$-fine and $\dot{P}$ is $\Gamma_\in$-fine, so that

$$|S((f\circ\phi).\varphi;\dot{P})-B|\le|S((f\circ\phi).\varphi;\dot{P})|+|S(f;\dot{P})-B|$$

$$\leq \in (b-a+2).$$

Since $\in > 0$ is arbitrary, we conclude that $(f \circ \phi).\varphi$ belongs to $R^*(I)$ with integral equal to B.

(b) If ϕ is increasing, then $\varphi \geq 0\, and\, so\, (f \circ \phi).\varphi)^{\pm} = (f^{\pm} \circ \phi).\varphi.$ We now apply part (a) to these functions and to $f^{\pm}$, using Theorem 7.11. Q.E.D.

The Second Substitution Theorem

We now give another version of the Second Substitution Theorem. It is possible to give a proof that is quite parallel to the proof of Theorem 12.5, but we prefer to deduce it from that result.

12.5 SECOND SUBSTITUTION THEOREM, II.

Let $I := [a,b]\, and\, J := [c,d]\, and\, let\, f : I \to R.$ be a continuous strictly monotone function with $\phi(I) \subseteq J$ and suppose that there exists a countable act $R \subseteq J$ such that $\phi(I) \subseteq J$ and suppose that there exists a countable set $C \subset J$ such that $\varphi(x) := \phi^{'}(x) \neq 0$ for all $x \in I - C.$ Let ψ be the continuous strictly monotone function inverse to ϕ so that

$$\psi(y) := \psi^{'}(y) = 1/\varphi(\psi(y)) \quad for \;\; y \in \phi(I-C),$$

And let $\psi(y) := 0\; for\; y \in \phi(C).$

(a) Then $f.\varphi$ belongs to $R^*(\phi(I))$ if and only if $f \circ \phi$ belongs to $R^*(I).$

(b) Also $f.\psi$ belongs to $L(\phi(I))$ if and only if $f \circ \phi$ belongs to $L(I).$

In either case, we have

$$(13.t) \qquad \int_{\phi(a)}^{\phi(b)} f.\psi = \int_a^b f \circ \phi.$$

Proof. (a) We note that $\psi(\phi(x)).\varphi(x) = 1\; for\; all\; x \in I - C.$ Therefore, if we let $f_1(y) := f(y).\psi(y)\; for\; y \in \phi(I),$ it follows that

$$(f_1 \circ \phi)(x).\varphi(x) = f(\phi(x)).\psi(\phi(x)).\varphi(x) = (f \circ \phi)(x)$$

For $x \in I - C.$ Theorem 12.5 implies that $f_1 = f.\psi$ belongs to $R^*(\phi(I))$ if and only if $(f_1 \circ \phi).\varphi = f \circ \phi$ belongs to $R^*(I)$ .In that case

$$\int_{\phi(a)}^{\phi(b)} f.\psi = \int_{\phi(a)}^{\phi(b)} f_1 = \int_a^b (f_1 \circ \phi).\varphi = \int_a^b f \circ \phi,$$

But this yields equation $(13.t)$

(c) If ϕ is increasing, then $(f.\psi)^{\pm} = f^{\pm}.\psi \; and \; (f \circ \phi)^{\pm} = f^{\pm} \circ \phi.$

Now apply part (a) Q.E.D

Another Substitution Theorem

We now obtain another version of the First Substitution Theorem in which we do not assume that ϕ is monotone, instead, we will assume that $\phi^{'}(x)$ exists and is $\neq 0$ except on a finite subset of $I.$

- 12.8 First Substitution Theorem, III. Let $I := [a,b] \, and \, J := [c,d] \, and \, let \; f : J \to R.$ Let $\phi : I \to R$ be continuous with $\phi : (I) \subseteq J$ and suppose that there exists a finite set $x \in I - E$ and we set $\varphi(x) := 0 \; for \; x \in E.$

(a) Then f belongs to $R^{*}(\phi(I))$ if and only if $(f \circ \phi).\varphi$ belongs to $R^{*}(I).$

(b) Also f belongs to $L(\phi(I))$ if and only if $(f \circ \phi).\varphi$ belongs to $L(I).$

In either case we have

$$(13.k) \qquad \int_{\phi(a)}^{\phi(b)} f = \int_0^b (f \circ \phi).\varphi.$$

Proof. (a) 0rder the points in $E \bigcup \{a,b\} \, by \, a =: \in_0 < \in_1 < ... < \in_m := b$ and let $I_k := [e_{k-1}, e_k] \; for \; k = 1,...,m.$ Then ϕ is continuous on each interval $I_k and \, \phi^{'}(x) \neq 0 \; for \; x \in (e_{k-1}, e_k).$ It follows from the Darboux intermediate value Theorem [B.S;p,174) that $\phi^{'}(x)$ does not change sign on $I_{k.}$ The Mean Value Theorem then implies that ϕ is strictly monotone on $I_{k.}$ Theorem 12.5 implies that f is integrable on $\phi(I_k)$ if and only if $(f \circ \phi).\varphi$ is integrable on $I_{k.}$ IT therefore follows from Theorem 3.7 and induction that f is integrable on $\phi(I)$ if and only if $(f \circ \phi).\varphi$ is integrable on $I,$ in which case the the additivity of the integral over subintervals implies that $(13.k)$ holds.

(b) The case of absolute integrability is handled similarly Q.E.D

Remark. We will not state a corresponding version of the Second Substitution Theorem, since it requires the existence of an inverse function in each of the finite subintervals. However it is clear that in certain instances, it might be useful to break the interval I into a finite number of parts and calculate the appropriate inverse functions on each part.

More Examples

12.9 Examples. (a) We saw in Example 6.12(a) that the Beta function

(13.λ) $B(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1}dx$

Exists when $p > 0, q > 0.$ If we introduce the substitution

$x = \phi(\theta) := (\sin\theta)^2 \; for\, \theta \in [0, \pi/2].$ an elementary calculation gives

(13.μ) $B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2q-1}(\cos\theta)^{2q-1}d\theta$

(b) Consider the integral $\int_\alpha^\beta \frac{\sin(\pi/x)}{x}dx \; where\, 0 < \alpha < \beta.$

Here we let $x = \phi(u) := 1/u \; for\, u > 0,$ nothing that ϕ is strictly decreasing and that noting that $\varphi(u) := \phi^{'}(u) = -1/u^2.$ Thus ϕ is an order-reversing map of the interval

$(1/\beta, 1/\alpha)^* onto [\alpha, \beta]. If\; we\, let\; f(x) := [\sin(\pi/x)]/x,$ then we have

$f \circ \phi(u) = [\sin(\pi u)]u \; and \; f \circ \phi(u).\varphi(u) = -[\sin(\pi u)]/u.$

If we apply formula (13.γ) and simplify, we obtain

(13.ν) $\int_\alpha^\beta \frac{\sin(\pi/x)}{x}dx = \int_{1/\beta}^{1/\alpha} \frac{\sin(ru)}{u}du,$

Which will be used in Example 12.10(b).

Infinite Integrals

In Section 16 we will discuss the integral of a function over infinite intervals, such as the interval $[\alpha, \infty],$ and we will obtain a version of Hake's Theorem asserting that the Riemann integral $\int_\alpha^\infty f$ exists if and only if $\int_a^c f$ exists for all $c > a$ and the limit $\lim_{c\to\infty} \int_a^c f$ exists.

We will now show that the substitution theorems we have established in this section sometimes give useful information concerning limits of this type. Hence, the results often provide an effective method of establishing the existence of, and of evaluating, these infinite integrals.

12.10 Examples. (a) If we take $x = \phi(u) := u/(1+u) \; for\, u \in [0, b]$ in equation (13.λ), we obtain (after an easy calculation):

$$\int_0^{b/(1+b)} x^{p-1}(1-x)^{q-1}dx = 2\int_0^b \frac{u^{p-1}du}{(1+u)^{p+q}}.$$

Since we have seen in Example 6.12 (a) that if $p>0, q>0,$ then the function $x \mapsto x^{p-1}(1-x)^{q-1}$ belongs to $R^*([0,1]),$ we conclude from Hake's theorem 12.8 and the fact that $b/(1+b) \to 1 \mathit{as}\ b \to \infty \mathit{that}$

$$B(p,q) = 2 \lim_{b\to\infty} \int_0^b \frac{u^{p-1}du}{(1+u)^{p+q}}.$$

By the version of Hake's Theorem to be proved in Section 16, we will have

$$(13.\xi) \quad B(p,q) = 2\int_0^\infty \frac{u^{p-1}du}{(1+u)^{p+q}}$$

(b) If we take $\beta = 1$ in formula $(13.v),$ we infer that

$$\int_0^1 \frac{\sin(\pi/x}{x}dx = \lim_{c\to\infty} \int_1^c \frac{\sin(\pi u)}{u}du.$$

Since $u \mapsto (1/u)\sin(\pi u)$ is bounded on [0,1]and continuous (if we define it to equal $\pi\, at\ x=0),$ there is no doubt about the existence of the integral $\int_0^1 (1/u)\sin(\pi u)du.$ Moreover, if we use $(13.v)$ with $\alpha = 1 < \beta,$ we obtain

$$\int_1^\beta \frac{\sin(\pi/x)}{x}dx = \int_1^{1/\alpha} \frac{\sin(\pi u)}{u}du.$$

Whence it follows that

$$\lim_{\beta\to\infty} \int_1^\beta \frac{\sin(\pi/x)}{x}dx = \int_0^1 \frac{\sin(\pi u)}{u}du.$$

Combining these observations, we obtain the nonobvious formula

$$(13.\circ) \qquad \int_0^\infty \frac{\sin(\pi/x)}{x}dx = \int_0^\infty \frac{\sin(\pi u)}{u}du.$$

Note that the existence of the integrals over $[1,\infty]$ on each side of $(13.\circ)$ follows from the existence of the integrals over (0,1] on the other side.

Exercises

Some of the following integrals are divergent, and others are convergent. When possible, evaluate the convergent integrals exactly, checking your result with the given approximate values. State which theorem(s) you use and identify the functions; you may also use Integration by Parts.

Assume that $0 \le a < b$.

12.A $(a)\int_0^3 x\sqrt{4+x^2}\,dx \approx 12.957,$ $(b)\int_{-1}^1 x\sqrt{4+x^2}\,dx.$

12.B $(a)\int_0^3 \frac{xdx}{1+x^2} \approx 1.151,$ $(b)\int_a^b \frac{x\,dx}{1+x^2}$

12.C $(a)\int_0^4 \frac{dx}{2+\sqrt{x}} \approx 1.227,$ $(b)\int_a^b \frac{dx}{2+\sqrt{x}}$

12.D $(a)\int_1^3 \frac{dx}{x\sqrt{x+1}} \approx 0.664,$ $(b)\int_0^3 \frac{dx}{x\sqrt{x+1}}$

12.E $(a)\int_1^5 x\sqrt{2x+3dx} \approx 37.498,$ $(b)\int_a^b x\sqrt{2x+3}\,dx.$

12.F $(a)\int_1^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx \approx 3.157,$ $(b)\int_0^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx.$

12.G $(a)\int_1^2 \frac{\sqrt{x-1}}{x}dx \approx 0.429,$ $(b)\int_0^2 \frac{\sqrt{x-1}}{x}\,dx.$

12.H $(b)\int_0^1 \frac{dx}{\sqrt{x-x^2}} \approx 1.571.$ $(b)\int_a^b \frac{\sqrt{x}}{1+\sqrt{x}}\,dx.$

12.I $(a)\int_0^4 \frac{dx}{\sqrt{x}(x+4)} \approx 0.785,$ $(b)\int_a^b \frac{dx}{\sqrt{x(x+4)}}.$

12.J $(a)\int_a^b \frac{\cos x\,dx}{\sqrt{x}(x+4)} = In(2+\sin x)\,|_a^b$ [Use Theorem 12.8]

12.K $(a)\int_a^b \frac{\cos x\,dx}{2-\cos^2 x} = Arc\tan(\sin x)\,|_a^b.$

12.L $(a)\int_3^8 \frac{dx}{x\sqrt{x+1}} \approx 0.405,$ $(b)\int_0^1 \frac{dx}{\sqrt{x-x^2}} \approx 1.571.$

Notes

12.N (a) $\int_0^1 \sqrt{\frac{x}{1-x^3}}dx \approx 1.047,$ (b) $\int_0^1 \frac{\sqrt{x}dx}{1+x^2} \approx 0.524.$

Check Your progress

1. Prove: Let $I := [a,b] and\ J := [c,d]$ and suppose that:

(i) $f : J \to R$ has a c-primitive $F\ on\ J.$

(ii) $\varphi : I \to R$ has a c-primitive ϕ on $I\ and \phi(I) \subseteq J.$

(iii) ϕ is a countable-to-one mapping of $I\ \mathrm{int}\, o\ J.$

Then $(f \circ \phi)\varphi \in R^*(I)\ and\ f \in R^*\left(\phi(I)\right) : moreover,$

$\int_a^b (f \circ \phi).\varphi = (F \circ \phi)|_a^b = \int_{\phi(\alpha)}^{\phi(b)} f.$

---

2. Prove: Let $f : J \to R$ and $\phi, \varphi : 1 \to R$ be as in Theorem 13.5. Given

$\in > 0,$ there exists an interval-gauge Δ_ϕ on I such that if $\dot{P} := \{(I_i, t_i)\}_{i=1}^n$

is any $\Delta_\phi - fine$ partition of I and if $\dot{P} := \{(\phi(I_i), \phi(t_i))\}_{i=1}^n$ then

$\left| S\left(f : \dot{P}\right) - S\left(f 0 \phi\right).\varphi; \dot{P}) \right| \leq \varepsilon\left(b - a + 1\right)$

3.Prove second substitution theorem.

12.6 LET US SUM UP

1. Let $I := [a,b] and\ J := [c,d]$ and suppose that:

$f : J \to R$ has a c-primitive $F\ on\ J.$

$\varphi : I \to R$ has a c-primitive ϕ on $I\, and \phi(I) \subseteq J$.

ϕ is a countable-to-one mapping of I into J.

Then $(f \circ \phi)\varphi \in R^*(I)\, and\; f \in R^*\left(\phi(I)\right) : moreover,$

$$\int_a^b (f \circ \phi).\varphi = (F \circ \phi)\,|_a^b = \int_{\phi(\alpha)}^{\phi(b)} f.$$

2. Let $f : J \to R$ and $\phi, \varphi : 1 \to R$ be as in Theorem 13.5. Given $\in > 0,$ there exists an interval-gauge Δ_ϕ on I such that if

$\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ is any $\Delta_\phi - fine$ partition of I and if

$\dot{P} := \{(\phi(I_i), \phi(t_i))\}_{i=1}^{n}$ then

$$\left| S\left(f : \dot{P} \right) - S\left(f\, 0\, \phi \right).\varphi; \dot{P}) \right| \leq \varepsilon\left(b - a + 1\right)$$

12.7 KEY WORDS

Interval-gauge

Substitution theorem

Riemann integral

12.8 QUESTIONS FOR REVIEW

1. Explain about First substation theorem-1
2. Explain about First substation theorem-II
3. Explain about First substation theorem-III
4. Explain about second substation theorem-1

12.9 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle
2. The elements of Integration and Lebesgue Meassure
3. A course on integration- Nicolas Lerner
4. General theory of Integration- Dr. E.W. Hobson

5. General theory of Integration- P.Muldowney

6. General theory of functions and Integration- Angus E.Taylor

12.10 ANSWERS TO CHECK YOUR PROGRESS

1. See section 12.2
2. See section 12.4
3. See section 12.5

UNIT-13 ABSOLUTE CONTINUITY

STRUCTURE

13.0 OBJECTIVE:

In this unit we will learn and understand about Absolute continuity, Properties of continuity, Lebesgues Differentiation and indefinite integrals of functions.

13.1 INTRODUCTION

A continuous function fails to be absolutely continuous if it fails to be uniformly continuous, which can happen if the domain of the function is not compact – examples are tan(x) over $[0, \pi/2)$, x^2 over the entire real line, and sin($1/x$) over $(0, 1]$. But a continuous function f can fail to be absolutely continuous even on a compact interval. It may not be "differentiable almost everywhere" (like the Weierstrass function, which is not differentiable anywhere). Or it may be differentiable almost everywhere and its derivative f' may be Lebesgue integrable, but the integral of f' differs from the increment of f (how much f changes over an interval). This happens for example with the Cantor function.

A finite measure μ on Borel subsets of the real line is absolutely continuous with respect to Lebesgue measure if and only if the point function $F(x) = (\mu(-\infty, x])$

is an absolutely continuous real function. More generally, a function is locally (meaning on every bounded interval) absolutely continuous if and only if its distributional derivative is a measure that is absolutely continuous with respect to the Lebesgue measure.

If absolute continuity holds then the Radon–Nikodym derivative of μ is equal almost everywhere to the derivative of F.

More generally, the measure μ is assumed to be locally finite (rather than finite) and $F(x)$ is defined as $\mu((0,x])$ for $x > 0$, 0 for $x = 0$, and $-\mu((x,0])$ for $x < 0$. In this case μ is the Lebesgue–Stieltjes measure generated by F. The relation between the two notions of absolute continuity still holds.

13.2 PROPERTIES OF CONTINUITY

- The sum and difference of two absolutely continuous functions are also absolutely continuous. If the two functions are defined on a bounded closed interval, then their product is also absolutely continuous.
- If an absolutely continuous function is defined on a bounded closed interval and is nowhere zero then its reciprocal is absolutely continuous.
- Every absolutely continuous function is uniformly continuous and, therefore, continuous. Every Lipschitz-continuous function is absolutely continuous.
- If f: $[a,b] \to \mathbf{R}$ is absolutely continuous, then it is of bounded variation on $[a,b]$.
- If f: $[a,b] \to \mathbf{R}$ is absolutely continuous, then it can be written as the difference of two monotonic nondecreasing absolutely continuous functions on $[a,b]$.

- If f: $[a,b] \to \mathbf{R}$ is absolutely continuous, then it has the Luzin N property (that is, for any such that , it holds that , where stands for the Lebesgue measure on **R**).
- f: $I \to \mathbf{R}$ is absolutely continuous if and only if it is continuous, is of bounded variation and has the Luzin N property.

Definition 7.3 we defined what it means for a function $F:[a,b] \to R$ to have bounded variation on $I := [a,b]$. We denoted the variation of $F\ by Var(F;I)$ and the collection of all functions having bounded variation on $I\ by BV(I)$. It was seen in Exercise 7.E that linear combinations and point wise products of functions in $BV(I)$. also belong to $BV(I)$. Also, it was noted in Exercise 7.J that a function is in $BV(I)$. if and only if it is the difference of two increasing functions. For that reason, in establishing results about functions in $BV(I)$, it is frequently useful to consider the case of increasing functions.

The following basic theorem was proved by Henri Lebesgue in 1904. A detailed proof of it is given in Appendix E.

13.3 LEBESGUES'S DIFFERENTIATION THEOREM.

If $F \in BV(I)$, then there exists a null set $Z \subset I$ such that the derivative $F^{'}(x)$ exists for all $x \in I - Z$.

In other words, a function $F\ in\ BV(I)$ is differentiable a.e. on I.

However, the converse assertion is not true; indeed, it was seen in Example 7.6(c) that the function $G(x) := \cos(\pi / x^2) x \in (0.1]\ and\ G(0) := 0$ is differentiable at every point of $I, but\ G \notin BV(I)$.

The question arises as to whether the derivative F' of a function $F \in BV(I)$ is integrable . The answer is: “Yes-but...”In fact, the derivative F' always belongs to $L(I)$; however, the integral of F' over $[a,x]$ may not yield F_0^x.

Theorem. If $F \in BV(I), then\, F' \in L(I).$ In addition. If F is increasing on $I := [a,b], then \int_a^x F' \leq F(x) - F(a) \quad for\, x \in I.$

Proof. It is enough to treat the case where F is increasing on $I.$ We extend F to $[a, b+1]$ by setting $F(x) := F(b)\ for\ x \in (b, b+1].$ If $n \in N,$ we define $f_n(x) := n\left[F(x+1/n) - F(x)\right] \quad for\ x \in I.$

Since F is increasing, it is measurable and f_n is also measurable and $f_n(x) \geq 0\ for\ x \in I.$ Also lebesgue's Theorem 14.1 implies that $\lim_n(f_n(x)$ exists a.e. and equals $F'(x).$ Moreover, we have that

$$(14.\beta) \qquad \int_a^b f_n = n\left[\int_a^b F(t+1/n)dt - \int_a^b F(t)dt.\right].$$

Theorem 3.21(a) or the First Substitution Theorem 13.1 imply that

$$\int_a^b F(t+1)/n)\,dt = \int_{a+1/n}^{b+1/n} F(t)dt = \int_b^{b+1/n} F(t)dt.$$

Consequently, equation $(14.\beta)$ becomes $\int_a^b f_n = n\left[\int_b^{b+1/n} F - \int_a^{a+1/n} F\right].$

But, $F(x) = F(b)\, for\, x \in \left[b, b+1/n\right] and\, F(a) \leq F(x)\ for\ x \in \left[a, a+1/n\right],$

whence $\int_a^b f_n \leq n\left[F(b).(1/n) - F(a).(1/n)\right] = F(b) - F(a).$

Since $f_n \geq 0,$ Faton's Lemma 8.7 implies that $F' \in L(I)\, and$

$$0 \leq \int_a^b F' \leq F(b) - F(a),$$

Which is $(14.\alpha)$ with $x = b.$ If we replace the interval [a,b] by the interval $[a, x]\ for\ x \in [a,b],$ we obtain $(14.\alpha)$ Q.E.D.

Examples. (a) The Cantor-Lebesgue singular function $\Delta : [0,1] \to R$ Theorem 4.17 is increasing and so belongs to BV([0.1]). Moreover. $\wedge'(x) = 0$ when $x \in [0,1] - \Gamma,$ where Γ is the Cantor set (see 4.15). Here we have $0 = \int_0^x \wedge' < \wedge(x) - \wedge(0)\, when\, x \in (0.1].$ Note also that $\wedge$ is continuous on $[0.1].$

(b) There exists a strictly increasing continuous function $f\ on [0.1] with\ f'(x) = 0\, a.e.$

We will now define the important class of functions that are "absolutely continuous". According to Hawkins [Hw-1;p.142ff.], Axel Harnack (1851-1888) called attention to this notion as early as the 1880s, and this property was also considered by other mathematicians near the end of the eighteenth century. However, the name was introduced in 1905 by Giuseppe Vitali (1875-1932).The reader will recall that we used the phrase "absolute continuity property" in connection with 10.10(a). The next definition is essentially this property for sets that are the union of a sub partition o f I (that is, a finite collection of nonoverlapping closed intervals in I).

Definition . Let $I := [a,b] and\, let\, F; I \to R.$ We say that F is absolutely continuous on I and write $F \in AC(I)$ if, for every $\in > 0$ there exists $n_\in > 0$ such that if $\{[u_j, \upsilon_j]\}_{j=1}^{s}$ is any subpartition of I such that

$$(14.\gamma) \qquad \sum_{j=1}^{s} |\upsilon_j - u_j| \le \eta_\in, \qquad then \qquad \sum_{j=1}^{s} |F(\upsilon_j) - F(u_j)| \le \in.$$

Note. It will be seen in an exercise that it is essential that the subintervals $\{[u_j, \upsilon_j]\}_{j=1}^{s}$ be nonoverlapping.

We now will establish some important properties of the class $AC(I)$.

Theorem. Let $I := [a,b]$ be a compact interval.

(a) If $F \in AC(I), then\, F$ is (uniformly) continuous on I.

(b) If $F \in AC(I), then\, F \in BV(I)$.

(c) If $F, G \in AC(I), and\ c \in R,$ then the functions

$cF, |F|, \ F+G, \qquad F-G, and\ F.G$

Also belong to $AC(I)$.

Proof. $If\, \xi \in I, given \in > 0, let\, \eta_\in > 0$ be as in Definition 14.4 If $s \in I\ and\, |s-\xi| \le \eta_\in.$ it follows that $|F(s) - F(\xi)| \le \in, so\, F$ is continuous at an arbitrary point $\xi \in I.$ Since $\eta_\in$ does not depend on ξ ,the function F is uniformly continuous on I .

(b) let $\eta_1 > 0$ be as in Definition 14.4 corresponding to $\in = 1.$ If J is any subinterval of

I with length $l(J) \le \eta_{1,}$ then Var $(F; J) \le 1.$ Now let $r \in N\, with\, r > (b-a)/\eta_1,$ and divided I int o r nonoverlapping intervals $I_{1,}..., I_r$ with length $(b-a)/r < \eta_1.$ Excercises 7.G and 7.H imply that

$$Var(F;I) = \sum_{k=1}^{r} Var(F;I_k) \le r.$$

(c) It is trivial that a constant multiple of $F \in AC(I)$ belongs to $AC(I)$.

Moreover, the inequality $\big\| F(\upsilon_j)| - F(u_j)\big| \le | F(\upsilon_j) - F(u_j)|,$ implies that $|F| \in AC(I).$ Further, since

$$\big|F \pm G)(\upsilon_j) - (F \pm G)(u_j) \le \big| F(\upsilon_j)| + |G(\upsilon_j) - G(\upsilon_j)|,$$

We readily conclude that $F \pm G \in AC(I).$

If $|F(x)|,|G(x)| \le M \; for \; x \in I,$ then since

$$|(FG)(\upsilon_j) - (FG)(u_j)| \le | F(\upsilon_j) - F(u_j)|.|G(\upsilon_j)|$$

$$+|F(u_j)|.|G(\upsilon_j) - G(u_j)|$$

$$\le M[| F(\upsilon_j) - F(u_j)| + |G(\upsilon_j)| - G(u_j)|],$$

It is seen that

$$\sum_{j=1}^{s} | FG(\upsilon_j) - FG(u_j) \le M\left[\sum_{j=1}^{s} | F(\upsilon_j) - F(u_j)| + \sum_{j=1}^{s} | G(\upsilon_j) - G(u_j)|\right],$$

Whence it follows that $FG \in AC(I).$ Q.E.D.

The reader should recall the notion of negligible variation introduced in Definition 5.11. We now show that a function in $AC(I)$ belongs to $NV_I(Z)$ for any null set $Z \subset I.$

Lemma. If $F \in AC(I) \, and \, Z \subset I$ is a null set, then $F \in NV_I(Z).\varepsilon > 0, let \, \eta_\epsilon > 0$ be as in Definition 14.4. Since is a null set, there exists a sequence $\{J_k\}_{k=1}^{\infty}$ of open intervals such that

$Z \subseteq \bigcup_{k=1}^{\infty} J_k$ and $\sum_{k=1}^{\infty} l(J_k) \le \eta_\epsilon . It \, t \in Z, let \, k(t)$ be the smallest index k such that $t \in J_k$ and choose $\delta_\epsilon(t) > 0$ such that $\left[t - \delta_\epsilon(t), t + \delta_\epsilon(t)\right] \subset J_{k(t).}$ Now let

$\dot{P}_0 := \{(I_j, t_j)\}_{j=1}^{\theta}$ *be a* $\{\delta_\in, \square\}$ -fine sub partition of I.*Then, for* $j = 1, ..., s,$

we have

$t_j \in \square$ and $I_j \subseteq | t_j - \delta_\in(t_j), t_j + \delta_\in(t_j)] \subseteq J_{k(t_j)}.$

Thus, for each $k \in N,$ the intervals in $\dot{P}_0$ is $\leq \eta_\in$ so that

$\sum_{j=1}^{0} | F(u_j) - F(u_j) | \leq \varepsilon.$ Since $\in > 0$ is arbitrary, this that F belongs

to $NV_t(\square).$ Q.E.D.

It has already been noted (see Theorem 4.17) that the Cantor-Lebesgue function $\wedge$ is continuous and in $BV(I),$ where $I := [0,1].$ However (see Exercise 5.P), it is not in $NV_I(T),$ where Γ is the Cantor set. Thus $\wedge$ is not in $AC(I).$

13.4 INDEFINITE INTEGRALS OF FUNCTIONS IN $L(I)$

It was seen in the Characterization Theorem 5.12 that F is an indefinite integral of a function in $R^*(I)$ if and only if F is differentiable a.e. and has negligible variation on its set of non differentiability. We now show that F is an indefinite integral of a function in $L(I)$ if and only if F is in $AC(I),$ or if and only if F belongs to $BV(I)$ and has negligible variation on its set of non differentiablity.

This result is sometimes called a “descriptive characterization” (or” descriptive definition”) of the Lebesgue integral, since it gives a necessary and sufficient condition for a function to be the indefinite integral of a function in $L(I)$.In contrast, the process of evaluating the integral as limits of sums is referred to as the “constructive definition “ of the integral.

Characterisation Theorem. Let $I : +[a,b]$ *and let* $F : I \to R$. Then the following assertions are equivalent.

F is an indefinite integral of a function is $L(I)$

Notes

$F \in AC(I).$

$F \in BV(I)\, and\, if\; RCI$ is the set where the derivative $F'(x)$ does not exist, then $F \in NV_I(Z).$

Proof. (a) $\Rightarrow (b)$ Suppose that $F(x) := C + \sqrt{\bar{a}} f$ for some $C \in R$ and some $f \in L(I)$. By Theorem 10.10(a), given $\in > 0$ there exists $\eta_{\in} > 0$ such that if $E \subseteq I$ is a measurable set with measure $|E| \leq \eta_{\in}$, then $f_{\in} |f| \leq \in$. It follows that if $\{I_j\}_{j=1}^{s} = \{(u_{j,} \upsilon_j)\}_{j=1}^{s}$ is any sub partition of I such that $\sum_{j=1}^{d} |(\upsilon_J - U_J| \leq \eta_{\in}\; and\; if\; \in := \bigcup_{j=1}^{s} I_j, then |E| \leq \eta_{\in}$. Therefore we have

$$(14.\delta) \sum_{j=1}^{s} |F(\upsilon_j) - F(\upsilon_j) = \sum_{j=1}^{s} |\int_{\upsilon_j}^{v_j} |f| =$$

Therefore $F \in AC(I)$

(b) $\Rightarrow (c)\, If\; F \in AC(I),$ then we have seen in Theorem 13.5 that F is continuous on $I\; and\; F \in BV(I).$ The Lebesgue Differentiation Theorem 13.1 imples that the set $R \subseteq I$ where $F'(x)$ does not exist is a null set. Lemma 13.6 now implies that $F \in NV_I(Z).$

(c) $\Rightarrow (a)\, Let\, R$ be the set of non differentiability of the function F and let $f(x) := F'(x)\; for\; x \in I - R\, and\; f(x) := 0\; for\; x \in R.$ The characterization Theorem 5.12 implies that

$f \in R^*(I)\, and\; F(x) - F(\alpha) = \int_a^x f\;\; for\, all\; x \in I,$ so that F is an indefinite integral of f. Q.E.D

The next corollary complements Lemma 13.6.

∘ **Corllary.** Let $I := [a,b]\, and\, let\, F : I \to R.$ Then $F \in AC(I)$ if and only if $F \in BV(I)$ and $F \in NV(R)$ for every null set $Z \subset I.$

Proof. $(\Rightarrow)$ This is consequence of Theorem 13.5(b) and Lemma 13.6.

$(\Rightarrow)$ This follows from Theorems 13.1 and $13.\tau.$ Q.E.D.

The next result shows that, for an increasing function F, the equality holds in $(13.\alpha)$ with $with\, x = b$ if and only if $F \in AC(I)$.

$\circ$ **13.9 Corollary.** Let $F : I \to R$ be increasing on $I = [a,b]$. Then $F \in AC(I)$ if and only if

$$(14.\varepsilon) \quad \int_a^b F' = F(b) - F(a).$$

Proof. $(\Rightarrow)$ If $F \in AC(I)$. then Theorem 13.7 implies that there exist $C \in R\, and\; f \in C(I)$ such that

$$F(x) = C + \int_0^a \quad for\, x \in I.$$

Therefore $C = F(a) and\, F(b) = F(a) + \int_a^b f$. The Differentiation .Theorem 5.0 implies that $f = F^{'} a.e.$ Therefore $\int_a^b f = \int_a^b F'$ and so $(14.\varepsilon)$ holds.

$(\Leftarrow)$ Suppose $(14.\varepsilon)$ holds and let $\xi \varepsilon (a,b)$ be arbitrary. We claim that

$$(14.\zeta) \qquad F(\xi) - F(a) = \int_0^\xi F'$$

If not, then Theorem 14.2 applied to the interval $[a,\xi]$ implies that

$$F(\xi) - F(a) > \int_0^\xi F'$$

If we apply Theorem 13.2 to the interval $[\xi . b]$, we infer that

$$F(b) - F(\xi) \geq \int_\xi^b F'$$

If we add the last two inequalities, we conclude that F(b) F(a) > $\int_a^b F'$, which contradicts $(14.\varepsilon)$. Therefore the equation $(14.\xi)$ holds for $\xi \in [a,b]$, so that F is an indefinite integral of $F' \in L([a,b])$. Therefore, Theorem 14.7 implies that $F \in AC(I)$. Q.E.D.

Note. It was seen in Example 4.18(a) that if Λ in the Cantor-Lebesque function, then $0 = \int_0^I \Lambda' C \Lambda(0) = 1.$ The proceeding corollary provides another proof of the fact that, although Λ is continuous and in $BV([a,b])$. It is not in AC[(0,1)].

Diagram 13.1 summarizes the inclusions between important classes of functions on a compact interval. In it, we denote the set of indefinite integrals of functions in R^* by $f\, R^*$ and shade that set. We denote the set of indefinite integrals of functions in $L\, by\, f\, L,$ the set of functions having bounded variation by $BV,$ the set of absolutely continuous functions by $AC,$ and the set of continuous functions by C In view of theorems $AC = f L = BV \bigcap f\, R^*.$

Singular Functions

We now introduce so important subset of the class of functions that see differentable a.e.

- **Definition**. A function $F : I \rightarrow R$ is said to be singular on I if its derivative $F(x) = 0$ *for a.e.* $x \in I.$

The Cantor-Lebesque singular function Λ considered in Theorem is singular in the sense just defined. Although Λ belongs to $BV([0,1]),$ we have seen that it does not belong to $AC([0,1]),$ and is not constant.

The next result is a basic one concerning singular functions and we will give two proofs of it. The first one is short, but depends on a number of deep theorems. Teh second proof, due to Gordon [G-4;pp.116-117] is longor, but entirely elementary.

Theorem. If $F \in AC(I)$ is singular on I ,then F is a constant function.

First Proof. Theorem 14.7 implies that $F(x) = F(a) + \int_a^z F'$ for all $x \xi I.$

Since $F'(x) = 0\, a.e.,$ we conclude that $F(x) = F(a)$ *forall* $x \in I.$

Second Proof. Since $F \in AC(I),$ given $\in > 0,$ there exists $\eta_\in > 0$ as in Definition .

Now let R be the set of all points $x \in I$ for which either $F^{'}(x)$ does not exist or $F^{'}(x) \neq 0.$ Since Ris a null set, there exists a sequence $(f_k)_{k=1}^{\infty}$ of open intervals containing R such that $\sum_{k=1}^{\infty} I(I_k) \leq \eta_{\in}.$

We now define a gauge $\delta_{\in} on I := [a,b]$ as follows.

(i) If $t \notin R,$ then $F'(t) = 0$ and the Straddle Lemma 4.4 implies that there exists $\delta_{\in}(t) > 0$ such that if $u, \upsilon \in I$ satisfy

$t - b_{\in}(t) \leq u \leq t \leq \upsilon \leq t + \delta_{\in}(t), then | F(\upsilon) - F(u) | \leq \varepsilon(\upsilon - u).$

(ii) If $t \in R,$ we let $k(t)$ be the smallest index k such that $t \in I_k$ and choose $\delta_{\in}(t) > 0$ so that $[t - \delta_{\in}(t), t + \delta_{\in}(t))] \subset I_{\eta(t).}$

Now let $\dot{P} = \{([x_{i-1,} x_i], t_i)\}_{i=1}^{n}$ be a $\delta_{\in}$-fine partition of I and consider the sets of indices.

$$S_i := \{i : t_i \notin R\} \quad and \quad S_z := \{i : t_t \in R\}.$$

If $i \in S_{i,}$ then we have $\backslash F(x_t) - F(x_i - 1) | \leq \varepsilon | x_1 - z_i - 1 |.$ Further, if $i \in S_z,$ then $[z_{i-1,} z_i] \subset I_{k(t,),}$ so that

$$\sum_{i \in E_z} (x_i - x_{i-2}) \leq \sum_{k=1}^{\infty} I(I_k) \leq \tau_{\in},$$

From which it follows that $\sum_{i \in S_z} | F(x_i) - F(x_i) - F(x_{i-1}) \leq \varepsilon.$ consequently, we have

$$| F(b) - F(a) | = \left| \sum_{i=1}^{n} | [F(x_i) - F(x_{i-1})] \right|$$

$$\leq \sum_{\in, S_i}^{n} | [F(x_i) - F(x_{i-1})] | + \sum_{i \in S_z} | F(x_i) - F(x_{i-1}) |$$

$$\leq \sum_{\in, S_i}^{n} \in (x_i - x_{i-1}) + \in \leq \in (b + c - 1).$$

Notes

Since $\in > 0$ is arbitrary, we conclude that $F(b) = F(a)$. But since the above argument applies to any subinterval $[a, x] \subseteq [a,b]$, we infer that $F(x) = F(a)$ *for all* $x \in [a,b]$.

Q.E.D

∘ **Lebesgue Decomposition Theorem.** If $F \in BV(I)$, then F can be respresented as the sum

(14.η) $F = F_a + F_{s,}$

Where $F_a \in AC(I)$ *and* $F_s \in BV(I)$ in singular on I. Moreover, this representation is unique up to a constant function.

Proof. We define F_a *and* F_a *for* $x \in I$ *by*

$$F_a(x) := \int_a^x F' \quad and \quad F_a(x) := F(x) - F_a(x).$$

Consequently, $F_s^{'} = F' - F_a^{'} = 0\, a.e. on\, I, so\, F_s$ is a singular function on I. Also, since F_a is the indefinite integral of $F' \in L(I)$ with base point 0,it follows from Theorem 13.7 that $F_a \in AC(I)$.

To establish the uniqueness, suppose that F also has the form $F = G_a + G_{s,}$ where $G_a \in AC(I)$ *and* $G_s \in BV(I)$ in singular on I. Then

$$F_a - G_a = G_a - F_s$$

So that $F_a - G_a$ is both absolutely continuous and singular on I Therefore, Theorem 13.11 implies that there exists a constant C such that $F_a = C_a + C and\ F_a = G_s - C$. Q.E.D

Check your Progress

1. Prove: If $F \in BV(I), then\, F' \in L(I)$. In addition. If F is increasing on $I := [a,b], then \quad \int_a^x F' \leq F(x) - F(a) \quad for\, x \in I.$

2. Prove: Let $I := [a,b]$ be a compact interval.

If $F \in AC(I), then\ F$ is (uniformly) continuous on I.

If $F \in AC(I), then\ F \in BV(I)$. If $F, G \in AC(I), and\ c \in R,$ then the functions $cF, |F|,\ F+G,\quad F-G, and\ F.G$ Also belong to $AC(I)$.

3. Prove: Let $I :+[a,b] and\ let\ F : I \to R$. Then the following assertions are equivalent. F is an indefinite integral of a function is $L(I)$

$F \in AC(I)$. $F \in BV(I)\ and\ if\ RCI$ is the set where the derivative $F'(x)$ does not exist, then $F \in NV_I(Z)$.

4. Prove: Let $F : I \to R$ be increasing on $I = [a,b]$. Then $F \in AC(I)$ if and only if $\int_a^b F' = F(b) - F(a)$.

5. Prove: If $F \in BV(I)$, then F can be represented as the sum

(14.η) $F = F_a + F_{s,}$ Where $F_a \in AC(I)\ and\ F_s \in BV(I)$ in singular on I.Moreover, this representation is unique up to a constant function.

13.5 LET US SUM UP

1. If $F \in BV(I), then\, F' \in L(I).$ In addition. If F is increasing on $I := [a,b], then\ \int_a^x F' \le F(x) - F(a) \quad for\ x \in I.$

2. Let $I := [a,b]$ be a compact interval. If $F \in AC(I), then\ F$ is (uniformly) continuous on $I.$ If $F \in AC(I), then\ F \in BV(I).$

If $F, G \in AC(I), and\ c \in R,$ then the functions

$cF, |F|,\ F+G, \quad F-G, and\ F.G$ Also belong to $AC(I).$

3. Let $I : +[a,b] and\ let\ F : I \to R$. Then the following assertions are equivalent. F is an indefinite integral of a function is $L(I) \quad F \in AC(I).$ $F \in BV(I)\, and\, if\ RCI$ is the set where the derivative $F'(x)$ does not exist, then $F \in NV_I(Z).$

4. Let $F : I \to R$ be increasing on $I = [a,b].$ Then $F \in AC(I)$ if and only if $\int_a^b F' = F(b) - F(a).$

5. If $F \in BV(I),$ then F can be represented as the sum

$F = F_a + F_{s,}$ Where $F_a \in AC(I)\, and\, F_s \in BV(I)$ in singular on $I.$

Moreover, this representation is unique up to a constant function.

13.6 KEY WORDS

Absolute continuity

Properties of continuity

Lebesgues Differentiation

Indefinite integral

13.7 QUESTIONS FOR REVIEW

1. Explain about properties of continuity
2. Explain about Lebesgues Differentiation
3. Explain about indefinite integrals of functions

13.8 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle

2. The elements of Integration and Lebesgue Meassure

3. A course on integration- Nicolas Lerner

4. General theory of Integration- Dr. E.W. Hobson

5. General theory of Integration- P.Muldowney

6. General theory of functions and Integration- Angus E.Taylor

13.9 ANSWERS TO CHECK YOUR PROGRESS

1. See section 13.3
2. See section 13.3
3. See section 13.4
4. See section 13.4
5. See section 13.4

.

UNIT-14 MAPPING PROPERTIES OF AC FUNCTIONS

STRUCTURE

14.0 OBJECTIVE

In this unit we will learn and understand about Pre-calculus integration, mapping properties and related definitions and theorems.

14.1 INTRODUCTION

We already examined exponential functions and logarithms in earlier chapters. However, we glossed over some key details in the previous discussions. For example, we did not study how to treat exponential functions with exponents that are irrational. The definition of the number e is another area where the previous development was somewhat incomplete. We now have the tools to deal with these concepts in a more mathematically rigorous way, and we do so in this section.

For purposes of this section, assume we have not yet defined the natural logarithm, the number ee, or any of the integration and differentiation formulas associated with these functions. By the end of the section, we will have studied these concepts in a mathematically rigorous way (and we will see they are consistent with the concepts we learned earlier). We begin the section by defining the natural logarithm in terms of an integral. This definition forms the foundation for the section. From this definition, we derive differentiation formulas, define the number ee, and expand these concepts to logarithms and exponentil functions of any base.

14.2 PRE-CALCULUS INTEGRATION

The first documented systematic technique capable of determining integrals is the method of exhaustion of the ancient Greek astronomer Eudoxus (*ca.* 370 BC), which sought to find areas and volumes by breaking them up into an infinite number of divisions for which the area or volume was known. This method was further developed and employed by Archimedes in the 3rd century BC and used to calculate areas for parabolas and an approximation to the area of a circle.

A similar method was independently developed in China around the 3rd century AD by Liu Hui, who used it to find the area of the circle. This method was later used in the 5th century by Chinese father-and-son mathematicians Zu Chongzhi and Zu Geng to find the volume of a sphere.

In the Middle East, Hasan Ibn al-Haytham, Latinized as Alhazen (c. 965 – c. 1040 CE) derived a formula for the sum of fourth powers. He used the results to carry out what would now be called an integration of this function, where the formulae for the sums of integral squares and fourth powers allowed him to calculate the volume of a paraboloid.[1]

The next significant advances in integral calculus did not begin to appear until the 17th century. At this time, the work of Cavalieri with his method of Indivisibles, and work by Fermat, began to lay the foundations of modern calculus, with Cavalieri computing the integrals of x^n up to degree $n = 9$ in Cavalieri's quadrature formula. Further steps were made in the early 17th century by Barrow and Torricelli, who provided the first hints of a connection between integration and differentiation. Barrow provided the first proof of the fundamental theorem of calculus. Wallis generalized Cavalieri's method, computing integrals of x to a general power, including negative powers and fractional powers.

Newton and Leibniz

The major advance in integration came in the 17th century with the independent discovery of the fundamental theorem of calculus by Leibniz and Newton. Leibniz published his work on calculus before Newton. The theorem demonstrates a connection between integration and differentiation. This connection, combined with the comparative ease of differentiation, can be exploited to calculate integrals. In particular, the fundamental theorem of calculus allows one to solve a much broader class of problems. Equal in importance is the comprehensive mathematical framework that both Leibniz and Newton developed. Given the name infinitesimal calculus, it allowed for precise analysis of functions within continuous domains. This framework eventually became modern calculus, whose notation for integrals is drawn directly from the work of Leibniz.

Formalization

While Newton and Leibniz provided a systematic approach to integration, their work lacked a degree of rigour. Bishop Berkeley memorably attacked the vanishing increments used by Newton, calling them "ghosts of departed quantities". Calculus acquired a firmer footing with the development of limits. Integration was first rigorously formalized, using limits, by Riemann. Although all bounded piecewise continuous functions are Riemann-integrable on a bounded interval, subsequently more general functions were considered—particularly in the context of Fourier analysis—to which Riemann's definition does not apply, and Lebesgue formulated a different definition of integral, founded in measure theory (a subfield of real analysis). Other definitions of integral, extending Riemann's and Lebesgue's approaches, were proposed. These approaches based on the real number system are the ones most common today, but alternative approaches exist, such as a definition of integral as the standard part of an infinite Riemann sum, based on the hyperreal number system.

14.3 APPLICATIONS

Integrals are used extensively in many areas of mathematics as well as in many other areas that rely on mathematics.

For example, in probability theory, integrals are used to determine the probability of some random variable falling within a certain range. Moreover, the integral under an entire probability density function must equal 1, which provides a test of whether a function with no negative values could be a density function or not.

Integrals can be used for computing the area of a two-dimensional region that has a curved boundary, as well as computing the volume of a three-dimensional object that has a curved boundary. The area of a two-dimensional region can be calculated using the aforementioned definite integral.

We will first show that functions in AC map null sets to null sets (and measurable sets to measurable sets). It will be seen in Theorem 14.15 that this property characterizes absolute continuity for a continuous function of bounded variation. In establish these results, we will need a few facts proved in Section 18.

14.4 THEOREMS RELATED TO PROPERTIES

- **14.13 Theorem.** Let $F \in AC(I)\, and\, let\, Z \subset I := [a,b]$ be a null set. Then $F(R)$ is a nulls set.

Proof Given $\varepsilon > 0, let\, \eta_\varepsilon > 0$ be as in Definition 14.4 and let $\{J_k\}_{k=1}^{\infty}$ be a contable collection of open intervals with

$$Z \subset \bigcup_{k \to 1}^{\infty} J_k \qquad and \qquad \sum_{k=1}^{\infty} t(J_k) \le \eta_\varepsilon.$$

Since the union $\bigcup_{j=1}^{\infty} J_k$ is an open set, it follows from $[B.S; p.315]$ that we may assume that the open intervals $J_k =: (u_k, \upsilon_k), k \in R,$ are pair wise disjoint. For each $k \in R,$ the Maximum-minimum theorem $[B.S; p.131]$ implies that there exist points a_k [respectively, b_k] in the compact interval $[u_k, \upsilon_k]$ where the restriction of F to this interval is minimized [respectively, maximized]. Therefore

$$(14.\theta) \quad F(Z) \subseteq \bigcup_{k=1}^{\infty} F(J_k) \subseteq \bigcup_{k=1}^{\infty} [F(a_k), F(b_k)]$$

For each $n \in R$ we have

$$\sum_{k=1}^{n} | b_k - a_k | \le \sum_{k=1}^{\infty} | \upsilon_k - u_k | \le \eta_\in.$$

Whence it follows that

$$\sum_{k=1}^{n} | F(b_{k)} - F(a_k) | \le \in.$$

But, since $n \in R$ is arbitrary, we conclude that

$$\sum_{k=1}^{\infty} | F(b_k) - F(a_k) | \le \varepsilon.$$

Therefore, it follows from $(14.\theta)$ that the set $F(R)$ is contained in the union of a countable collection of (closed) intervals with total length $\le \varepsilon.$ Consequently, $F(R)$ is a null set. Q.E.D.

Remark. The condition that F sends null sets to null sets was called Condition (N) by Luzin by(1915).

14.14 Theorem. *If* $F \in AC(I)$ and if $E \subseteq I := [a,b]$ is a measurable set, then $F(E)$ is a measurable set.

Proof. The proof is based on the fact (see Theorem 18.19) that if $E \subseteq I$ is a measurable set, then there exists a null set Z and a sequence $(K_n)_{n=1}^{\infty}$ of compact sets in I such that $E = ZU\bigcup_{n=1}^{\infty} K_n$. Since K_n is compact, then its image $F(K_n)$ is also compact and therefore (see

Theorem 18.13) is measurable. By Theorem 14.13, the set $F(\square)$ is a null set and hence is measurable. Since

$$F(E)=F(Z)U\bigcup_{n=1}^{\infty}F(K_n),$$

Theorem 10.2(a) implies that $F(E)$ is measurable Q.E.D.

We now state a theorem that was proved in 1925, independently, by Stefan Banach (1892-1945) and M.A. Zarecki. We will give a proof only for the case of an increasing function . The general case is treated in [He, ST; pp.288ff.,303-304]and |G-3;p.99].

14.55 **Theorem** (Banach-Zarecki).If F is continuous on $I:=[a,b], if\ F$ belongs to $BV(I)$ and if F sends null sets to null sets, then $F\in AC(I)$.

Proof. Suppose that $F\notin AC(I),$ so that there exists $\varepsilon>0$ such that for each $n\in\square$ there is a subpartition of I into compact intervals $I_{km}:=[a_{kn},b_{kn}], k=1,...,s(n),$ with

$$(14.\iota)\quad \sum_{k=1}^{s(n)}(b_{km}-a_{km})\leq 1/2^n.$$

But such that

$$(14.k)\quad \sum_{k=1}^{s(n)}|F(b_{km})-F(a_{kn})|\geq \varepsilon.$$

We now define $E_m:=\bigcup_{n=m}^{\infty}\bigcup_{k=1}^{s(n)}I_{km},$ so that $E_m,$ being a countable union of compact intervals, is a measurable set. By inequality (10.ξ), we have

$$|E_m|\leq\sum_{n=m}^{\infty}\sum_{k=1}^{s(n)}(b_{kn}-a_{kn})\leq\sum_{n=m}^{\infty}1/2^n=1/2^{m-1}$$

Therefore, we conclude that $\lim_m|E_m|=0.$

Since F is continuous, then $F(E_m)$ is also a countable union of compact intervals and so is measurable, Since $I\supseteq E_m\supseteq E_{m+1}\supseteq...,$ formula $(10.\delta)F, the\ set\ F(E)$ is a null set. But, since $F(I)\supseteq F(E_m)\supseteq F(E_{m+1})\supseteq...,$ another application o f $(10.\delta)$ implies that

$$(14.\lambda)\quad \lim_{m\to\infty}|F(E_m)|=|F(E)|=0.$$

Notes

On the other hand, if F is assumed to be increasing, we have $F(I_{km}) = |F(a_{km}), F(b_{km})]$, whence it follows from $(14.k)$ that

$$|F(E_m)| \geq \sum_{k=1}^{s(m)} |F(b_{km}) - F(a_{km})| \geq \varepsilon.$$

Which contradicts. $(14.\lambda)$.Therefore, we conclude that $F \in AC(I)$.

Q.E.D.

Integration by Parts

We now given in Integration by Parts Theorem for Lebesgue integrable functions.

14.16 Theorem. If $I := [a,b] and\ if\ F, G \in AC(I), then$

$$(14.\mu) \quad FG|_a^b = \int_a^b F'G + \int_a^b FG'.$$

Proof. Since $F, G \in AC(I),$ there exists a null set $Z \subset I$ such that $(FG)'(x) = F'(x)G(x) + F(x)G'(x) \quad for\ x \in I - \square$.

Since $F' \in\!: (I)\ and\ G$ is continuous (and hence is bounded and measurable), then $F'G \in L(I)$. Similarly. $FG' \in L(I)$, so that

$$\int_a^b (F'G + FG') = \int_a^b F'G + \int_a^b FG'.$$

By Theorem 14.5(c), the product $FG \in AC(I),$ whence it follows from Theorem 14.7 that

$$\int_a^b (FG)' = FG|_a^b$$

Consequently $(14.\mu)$ follows from these formulas. Q.E.D.

The reader should compare this result with Theorems 12.2 and 12.3

Substitution Theorems

We return again to the topic of the substitution theorems. In Theorem 13.5, we required that the substitution function ϕ be strictly monotone and have a derivative except on a countable set. Here we will consider the case where ϕ is absolutely continuous (and so may fail to have a derivative on a null set). However, we will now require that the functions $f = F' or (f \circ \phi).\varphi$ are lebesgue intergrable.

We will stae a theorem due to James B. Serrin and Dale E.Varberg. We refer the reader to their paper $[S.V]$ or to [St;p.325] for a detailed proof.

14.17 Substitution Theorem. Let $I := [a,b] and\ J := [c,d]$ and let $\phi \in AC(I), \phi(I) \subseteq J\ and\ F \in AC(J)$.Further, let $\varphi\ and\ f$ be the derivatives of ϕ *and* F, respectively, when these derivatives exist, and equal to 0 else where.

Then the following statements are equivalent:

(a) $F \circ \phi$ belongs to $AC(I)$.

(b) $(f \circ \phi).\varphi$ belongs to $L(I)$ and

$$(14.v) \quad \int_{\phi(a)}^{\phi(t)} f = \int_{s}^{t} (f \circ \phi).\varphi \quad for\, all\, s,t \in I.$$

14.18 Remarks. (a) It is quite possible that F and ϕ are absolutely continuous, but their composition $f \circ \phi$ is not absolutely continuous.

For example, let

$F(x) := \sqrt{x}$ *for* $x \in [0,1]$, *and* $\phi(x) : -(x \sin x^{-1})^2$ *for* $x \in (0.1]$ and

$\phi(0) := 0.$ Then both F *and* ϕ are absolutely continuous, but $F \circ \phi$ does not have bounded variation n [0.1], so it is not absolutely continuous on this interval.

(b) If $\phi \in AC(I)$ is monotone on I *and if* $F \in AC(J)$, then it is an exercise to show that $F \circ \phi$ is also absolutely continuous. Therefore, $(14.v)$ holds when $f \in L(J)$ *and* $\phi \in AC(I)$ is monotone.

(c) if $\phi \in AC(I)$ and if F satisfies a Lipschitz condition on I (that is, there exists a constant $K > 0$ such that $F \circ \phi$ is absolutely continuous o

I

(d) If $f \in L(J)$ is bounded on J, then it is an exercise to show that its indefinite integral F satisfies a Lipschitz condition on J. Thus, $(14.v)$ holds when $f \in L(J)$ is bounded and $\phi \in AC(I)$.

Other instances in which $(14.v)$ holds are given in the next result, which is proved in [St;p.326].

14.19 Theorem. Let $\phi \in AC(I)$ *and* $f \in L(J)$. Then equation $(14.v)$ holds if any one of the following conditions hold:

(a) ϕ is monotone on I.

(b) f is bounded on J.

(c) $(f \circ \phi).\varphi$ belongs to $L(I)$.

It is worth nothing that the indeed, if F and ϕ are as in Remark 14.18(a), then $f = F'$ belongs to $L(J)$; however, the function $(f \circ \phi).\varphi$ does not belong to $L(I)$.

14.5 MORE MAPPING PROPERTIES

We will now show that the indefinite integrals of functions in $R*(I)$, although not necessarily in $AC(I)$. have the same mapping property for null sets. This theorem was known for the indefinite integral of a Denjoy-Perron integrable function, and hence also for generalized Riemann integrable function. A direct proof was given by Xu Dongfu and Lu Shipan [X,L].

- **14.20 Theorem.** If F is an indefinite integral of $f \in R^*(I)$. then F sends null sets to null sets.

Proof. Let $Z \subset I :-[a,b]$ be a null set and let

$f_Z(x) := f(x)\ for\ x \in I - Z\ and\ f_Z(x) := 0\ for\ x \in Z$.Since

$f \in R^*(I)\ and\ Z$ is a null set, then $f_Z \in R^*(I)\ and\ F$ is also the indefinite integral of f_Z.

Let $\in > 0$ be given and let $\delta_\in$ be a guage as in Definition 1.7 for f_Z.

The family $F := \{[x-r, x+r] : x \in Z, 0 < r \le \delta_\in(x)\}$ is a Vitali covering of Z. Since F is continuous, the family F' consisting of the nondegenerate intervals in $\{F(J) : J \in F\}$ is s vitali covering of $F(Z)$ .Thus, by the Vitali Covering Theorem 5.8, there exist disjoint intervals $F'(I_1), i = 1, ..., p,$ from F' and closed intervals $\{J_i : i \ge p+1\}\ in\ F(I)$ such that

$$(14.\xi) \qquad F(Z) \subseteq \bigcup_{i=1}^{p} F(I_i) U \bigcup_{i=p+1}^{\infty} J_i \qquad with \sum_{ip+1}^{\infty} I(J_i) \le \varepsilon$$

Since F is continuous, there exist points $a_i, b_i \in I_i$ such that $FI_i = [F(a_i), F(b_i))]$. Let x_i be the midpoint of $I_i \in F$ so that $z, \in Z$, and choose $c_i \in \{a, b\}\ so\ that$

$$| F(x_i) - F(c_i) \ge | \frac{1}{2} | F(b_i) - F(a_i) | = \frac{1}{2} \iota(F(I_i)).$$

For $i = 1, ..., p.$ we let J_i be the interval with endpoints $x_i\ and\ c_i$, lagged by x_i .Since the $F(I_i)$ are dispoint, so are the I_i and hence also the J_i

.Thus $\dot{P} := \{(J_i, x_i)\}_{i=1}^{P}$ is a $\delta_\varepsilon - fine$ sub partition of I. Since $f_Z(x_i) = 0$, it follows from Corollary 5.4 of the Saks-Henstok Lemma, applied to

f_Z *and* $\dot{P}$ that $\sum_{i=1}^{p} I(F(I_i)) \leq 2\sum_{i=1}^{p} |F(t_i) - F(c_i)| \leq 4\varepsilon.$

Using (14.ξ), we deduce that $F(Z)$ is contained in the union of a countable collection of closed intervals with total length $\leq 5\varepsilon$. Since $\varepsilon > 0.$ is arbitrary we conclude that we conclude that $F(Z)$ is a null set.

Generallized AC

In the study of the Denjoy and Perron integrals, extensive use is made of classes of functions having bounded variation or absolute continuiry in a variety of generalized senses. The standard reference for this material is the book of Sake [S.2; especially Chapters 7 and 8]. Recently, Gordon [G-3] has given a through and modern treatment of this theory, which he relates to the generalized Riemann (Henstock-Kurzwell) integral that we have been discussion Gordons' treatment is very lucid, but it is inevitably complicated by the multiplicity of these generalized classes and their interrelations. While we will not go into these matters, we do wish to state another

characterization of the indefinite integrals of functions in $R^*(I)$, where $I := [a,b]$.

The next definition is taken from , [G-3;p,146].

14.21 Definition. (a) If $E \subseteq I$, we say that $F : I \to R$ belongs to the class $AC_\delta(E)$ if for every $\in > 0$, there exist $\eta_\in > 0$ and a gauge $\delta_\in$ *on* E such that if $\{(|u_i, \upsilon_i], t_i)\}_{t=1}^{s}$ is a $\delta_\varepsilon, E) -$ *fine* subpartition of E such that

$\sum_{i=1}^{s} |\upsilon_i - u_i| \leq \tau_{/\varepsilon}$, then $\sum_{i=1}^{s} |F(\upsilon_i) - F(u_i)| \leq \tau_{/\varepsilon}$,

(b) We say that F belongs to the class $ACG_\delta(I)$ if there exists a sequence $(F_m)_{n=1}^{\infty}$ of sets in I such that $I = \bigcup_{n=1}^{\infty} E_n$ *and* $F \in AC_\delta(E_n)$ for each $n \in N.$

The reader will note that the definition of $AC_\delta(E)$ contains the ingredients of both of the classes $AC(I)$ *and* $NV_I(E)$. In this connection, Gordon $[G.3; p.147]$ proved the following theorem, which is clearly related to our Characterization Theorem 5.12

Notes

14.22 Theorem. A function f belongs to $R^*(I)$ if and only if there exists a function $F \in ACG_\delta(I)$ such that $F' = f$ *a.e.*

In order to establish that the generalized Riemann integral coincides with the Denjoy and perron integrals, Gordon shows that the class $ACG_\delta(I)$ coincides with a class $ACG_\delta(I)$, which affords the simplest treatment of the Denjoy integral. For the record, we will give a definition of this class of functions.

First, we define the oscilliation of a bounded function F *on a set* $A \subseteq I$ by

$\omega p(A) := \sup, \{[u_i, \upsilon_i]\}_{i=1}^{s}$ is a collection of non overlapping intervals with endpoints in E *and such that*

$\sum_{i=1}^{s} |\upsilon_i - \upsilon_i| \leq \eta_e, then \sum_{i=1}^{i} ([\upsilon_i, \upsilon_i)] \leq \varepsilon,$ Finally, we say that

$F' \in ACG_{\bullet}(I) if\ F'$ is continuous on I and there is a countable collection $(E_n)_{n=1}^{\infty}$ of sets in I with $E = \bigcup_{i=1}^{\infty} and\ F \in AC_{\bullet}(E_n)\ for\ n \in$ □ .

Exercises

14.A A function $F : I \to R$ is said to satisfy a Lipschitz condition on $I := [a, b]$ if there exists a constant $M > 0$ such that $|F(x) - F(y)| \leq M\,|x - y|$ for all $x, y \in I$. Prove that such an F belongs to $AC(I)$.

14.B If $F : I \to R$ satisfies $|F'(x)| \leq M$ *for all* $|x \in I$, show that F satisfies a Lipschitz condition on I, and therefore belongs to $AC(I)$.

14.C Let $F : I \to R$. Show that $F \in AC(I)$ if and only if for every $\in > 0$ there exists $\zeta_\in > 0$ such that if $\{[u_j, \upsilon_j]\}_{j=1}^{s}$ in any subpartition of I satisfying $\sum_{j=1}^{s} (\upsilon_j - u_j) \leq \zeta_{\varepsilon,}$ then $\sum_{j=1}^{s} (F(\upsilon_j) - F(u_j)) \leq \varepsilon$.

14.D Let $F : I \to R$ Show that $F \in AC(I)$ if and only if for every $\in > 0$ there exists $\theta_\in > 0$ such that if $\{[u_j, \upsilon_j]\}_{j=1}^{\infty}$ is any sequence of non overlapping intervals satisfying $\sum_{j=1}^{\infty} |\upsilon_j - u_j| \leq \theta,$ then $\sum_{j=1}^{\infty} |F(\upsilon_j) - F(u_j)| \leq \varepsilon$.

14.E (a) Let $S(x) := \sqrt{x}$ on the interval $I_{\in} := [c,b] with \in > 0.$ Show that S satisfies $|S(x) - S(y)| \leq (1/2\sqrt{c)}\,|x - y|$ for all $x, y \in I_{c,}$ so that $S \in AC(I_c).$

(b) show that S does not satisfy a Lipschitz condition on $I_0 = [0, b].$

(c) Prove that $S \in AC(I_c).$ [Hint: Given $\in > 0$,let $c > 0$ be sufficiently small, and then break $\sum_{j=1}^{s} |S(\upsilon_j) - S(u_j)|$ into a sum of intervals in $[0, c]$ plus intervals in $[c, b]$

14.F Show that the function $S(x) := \sqrt{x}$ AC([0,1]) has the property that for any $\eta > 0$ these exists a finite collection of intervals $\{[u_j, \upsilon_j]\}_{j=1}^{s}$ satisfying $\sum_{j=1}^{s} |(\upsilon_j) - S(u_j)| \leq \eta and \sum_{j=1}^{s} |S(\upsilon_j) - S(u_j)| \geq 1.$

14.G Supose F is a continuous monotone function on [c,b]. If $F \in AC([c, b])$ for every $c \in (a, b].$ prove that $F \in AC([a, b]).$

14.H If $F :\to R$ is a continuous function in $BV(I)$ and $F'(x)$ exists except on a countable asset, prove that $F \in AC(I).$

Check Your Progress

1. Prove: Let $F \in AC(I) and\ let\ Z \subset I := [a, b]$ be a null set. Then $F(R)$ is a nulls set.

____

____

____

2. Prove: $If\ F \in AC(I)$ and if $E \subseteq I := [a, b]$ is a measurable set, then $F(E)$ is a measurable set.

____

____

____

3. Prove: If F is continuous on $I := [a, b], if\ F$ belongs to $BV(I)$ and if F sends null sets to null sets, then $F \in AC(I).$

____

____

____

4. Prove: If F is an indefinite integral of $f \in R^{*}(I)$. then F sends null sets to null sets.

____

____

____

14.6 LET US SUM UP

1. Let $F \in AC(I)\, and\, let\, Z \subset I := [a,b]$ be a null set. Then $F(R)$ is a nulls set.

2. $If\ F \in AC(I)$ and if $E \subseteq I := [a,b]$ is a measurable set, then $F(E)$ is a measurable set.

3. If F is continuous on $I := [a,b], if\ F$ belongs to $BV(I)$ and if F sends null sets to null sets, then $F \in AC(I)$.

4. If $I := [a,b] and\ if\ F, G \in AC(I), then$

(14.μ) $FG\,|_a^b = \int_a^b F'G + \int_a^b FG'.$

5. Let $I := [a,b] and\ J := [c,d]$ and let $\phi \in AC(I), \phi(I) \subseteq J\ and\ F \in AC(J)$. Further, let $\varphi\, and\ f$ be the derivatives of $\phi\, and\ F$, respectively, when these derivatives exist, and equal to 0 else where.

Then the following statements are equivalent:

$F \circ \phi$ belongs to $AC(I)$.

$(f \circ \phi).\varphi$ belongs to $L(I)$ and

$\int_{\phi(a)}^{\phi(t)} f = \int_s^t (f \circ \phi).\varphi \quad for\ all\ s,t \in I.$

6. Let $\phi \in AC(I)\, and\ f \in L(J)$. Then equation $(14.v)$ holds if any one of the following conditions hold:

(a) ϕ is monotone on I.

(b) f is bounded on J.

(c) $(f \circ \phi).\varphi$ belongs to $L(I)$.

7. If F is an indefinite integral of $f \in R^{*}(I)$. then F sends null sets to null sets.

8. If $E \subseteq I$, we say that $F : I \rightarrow R$ belongs to the class $AC_{\delta}(E)$ if for every $\in > 0$, there exist $\eta_{\in} > 0$ and a gauge $\delta_{\in} on E$ such that if $\{(|u_i, \upsilon_i], t_i)\}_{t=1}^{s}$ is a $\delta_{\varepsilon}, E) - fine$ subpartition of E such that $\sum_{i=1}^{s} | \upsilon_i - u_i | \leq \tau_{/\varepsilon}$, then $\sum_{i=1}^{s} | F(\upsilon_i) - F(u_i) | \leq \tau_{/\varepsilon}$,

14.7 KEY WORDS

Bounded on

Null sets

Sub partitions

Gauge- Integration

Riemann Integration

Mapping properties

14.8 QUESTIONS FOR REVIEW

1.Explain about definitions related to mapping properties of integration.

2. Prove: Let $F \in AC(I)$ *and let* $Z \subset I := [a,b]$ be a null set. Then $F(R)$ is a nulls set.

3. Prove: *If* $F \in AC(I)$ and if $E \subseteq I := [a,b]$ is a measurable set, then $F(E)$ is a measurable set.

14.9 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle

2. The elements of Integration and Lebesgue Meassure

3. A course on integration- Nicolas Lerner

4. General theory of Integration- Dr. E.W. Hobson

5. General theory of Integration- P.Muldowney

6. General theory of functions and Integration- Angus E.Taylor

14.10 ANSWERS TO CHECK YOUR PROGRESS

1. See section 14.4
2. See section 14.4
3. See section 14.5
4. See section 14.5